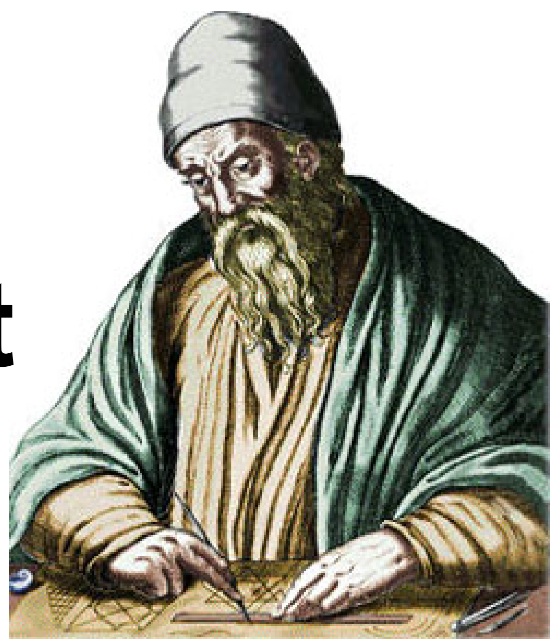
The Axioms That Euclid Forgot



The Plan

- David Hilbert's axioms for Euclidean Geometry

  Outline of Consistency and Independence

- Three Proofs of the Sylvester-Galli theorem

  For a set of points on the plane, either all are in a line or there is a line containing just two.

Euclid's Common notions

1. Things that are equal to the same thing are equal to one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things that coincide with one another are equal to one another.
5. The whole is greater than the part.

Euclids Axioms

1. To draw a straight line between two points.
2. To extend a finite straight line continuously.
3. To describe a circle with any centre and radius.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The Foundations of Geometry

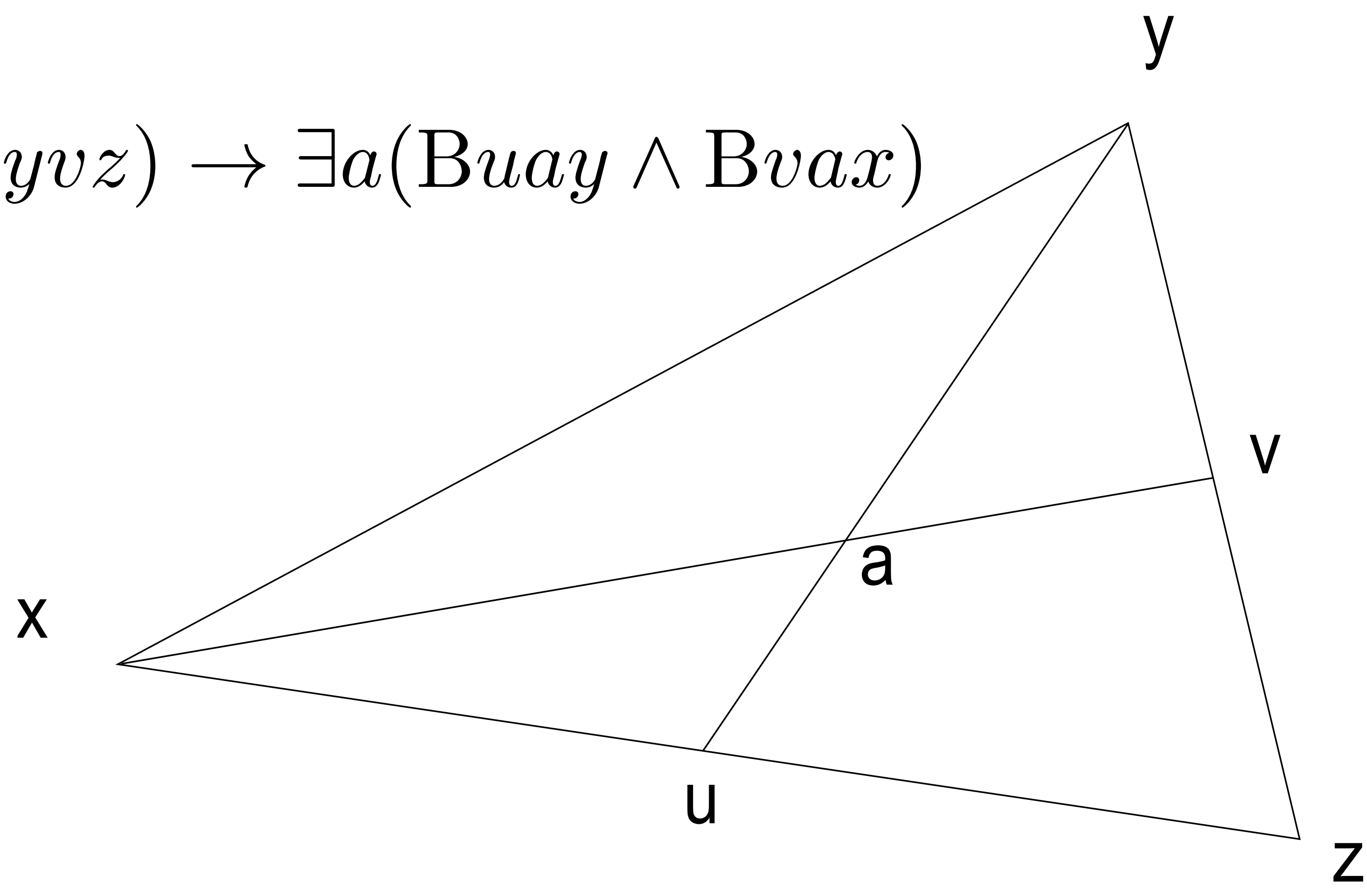
David Hilbert lectures1898-1899

Wanted the
Axiomatization of
all Mathematics

Tarski's version

- Tarski in 1920 devised axioms in first order logic with statements like Pasch's Axiom:

$$(\mathrm{B}xuz \wedge \mathrm{B}yvz) \rightarrow \exists a(\mathrm{B}uay \wedge \mathrm{B}vax)$$



Five sets of axioms

- 7 axioms of connection – points lines planes and space
- 4 axioms of order
- Euclid's axiom of parallels
- 6 axioms of congruence – lines and angles
- Archimedean axiom of continuity

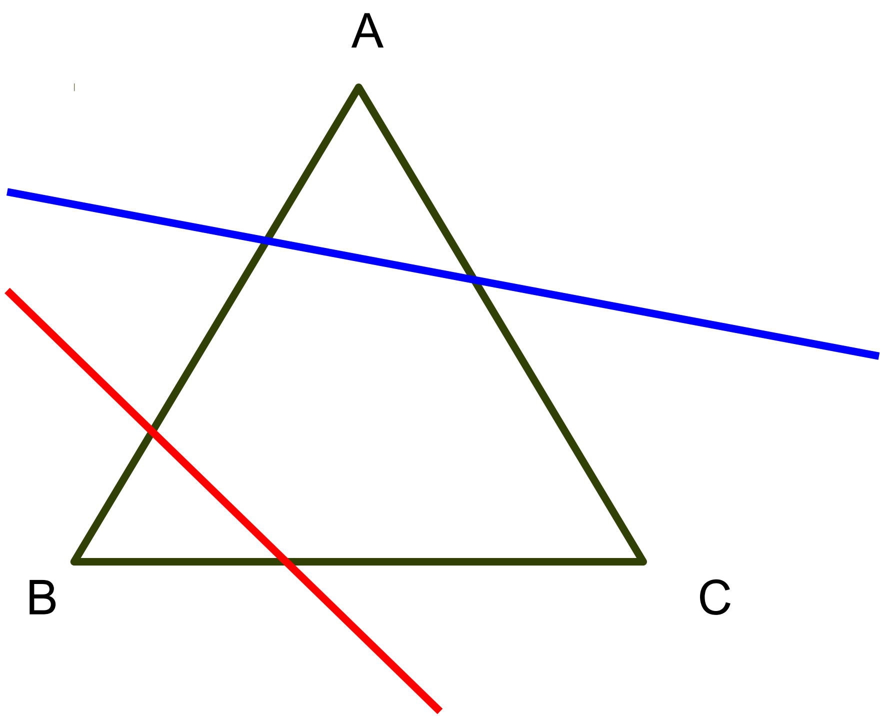
  + Axiom of completeness

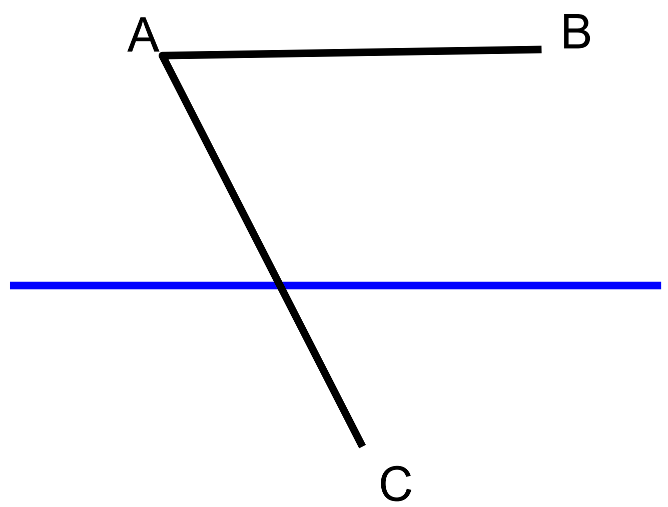
Axioms of Order

- [ABC] – B is between A and C

  Segment AC is all points between A and C

  Pasch's Axiom. Concept of a side of a line





Pasch's Theorem

Stated by Moritz Pasch 1882 along with Pasch's Axiom.

Cannot be derived using Euclid's axioms.

- Given points A, B, C, and D on a line, if B is between A and C, and C is between B and D, then B is between A and D.
- Was included as an axiom by Hilbert but E.H. Moore and R.L. Moore independently proved that this axiom is redundant in 1902.

Euclid's Axiom of Parallels

- Playfair's Axiom: In a plane for every line and a point not on the line there is exactly on line through the point that does not meet the original line.

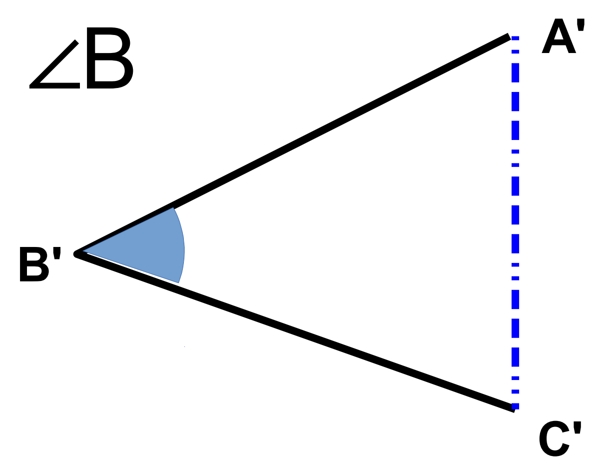
Axioms of Congruence

- 3 axioms Congruence of segments

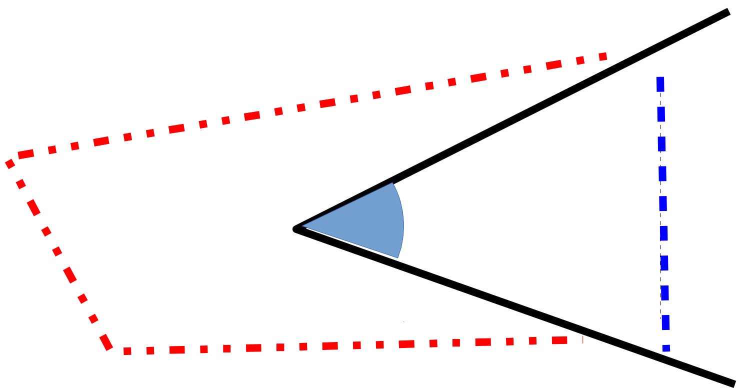
  Two rays (half lines) define an angle
- Axiom can always find a congruent angle on a ray on a given side.≡
- A'B' ≡ AB B'C' ≡ BC ∠B' ≡ ∠B

  implies ∠A' ≡ ∠A

  and   ∠C' ≡ ∠C



- Inside and outside of an angle

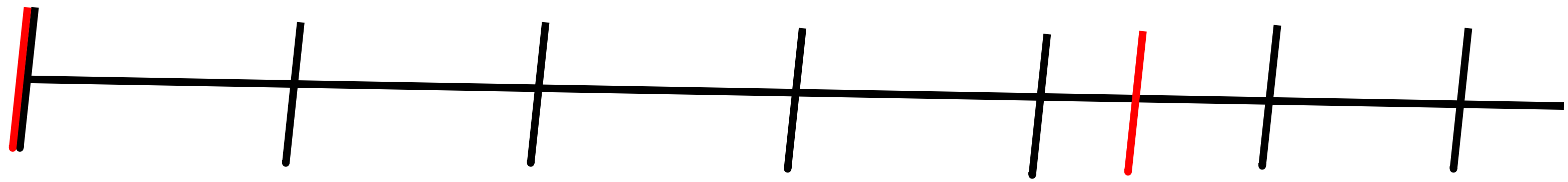


- Instead of being an axiom it is a theorem that all right angles are the same

  Also a circle is a defined construct rather than being an axiom.

Axiom of Archimedes

- One can put a finite number of congruent segments together to exceed any given segment.



- Can show this is needed with the model $2x^2 + 3x + 1 > N(x+2)$ for large enough x
- Axiom of Line Completeness – not needed for getting a model with constructible points.

Consistency of the Axioms

- Compatibility is tested by finding a model where all th axioms are true.

  Easily done by mapping points to (x, y, z) where x, y, z are real – or more restricted where they are from the field generated from 1 by addition, subtraction, multiplication, division and $\sqrt{1+x^2}$

  Then check all the axioms (except completeness) are satisfied.

Independence

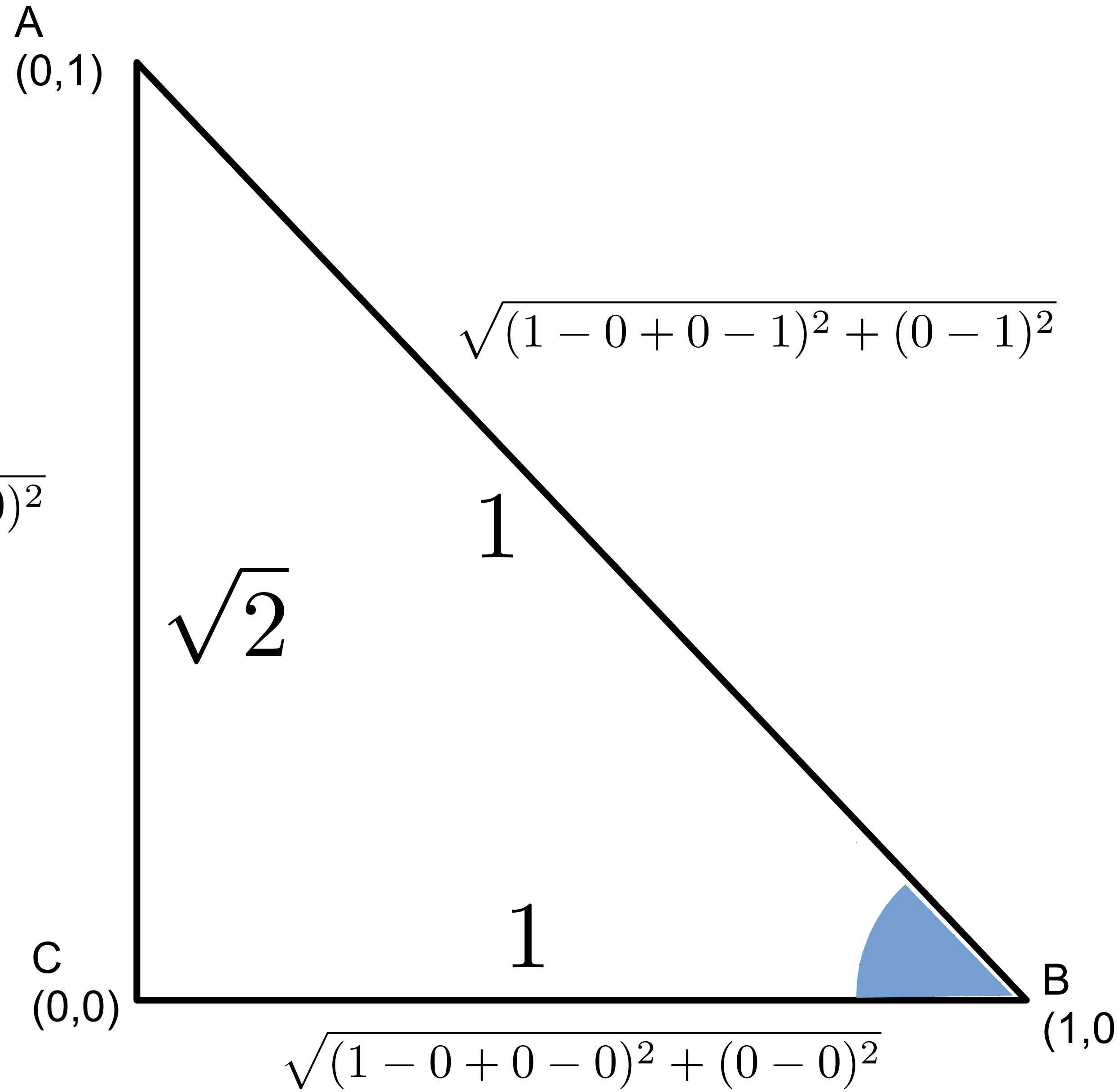
- For each axiom need a mode where it is false and all the others true.

  e.g. for axiom of congruence of triangles

  Like normal descarian geometry and all the angles the same.

  Distance defined by

$$\sqrt{(x_1 - x_2 + y_1 - y_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$



$\sqrt{(0-0+1-0)^2+(1-0)^2}$

BA = BC

but

$\angle$ BAC ≠ $\angle$ BCA

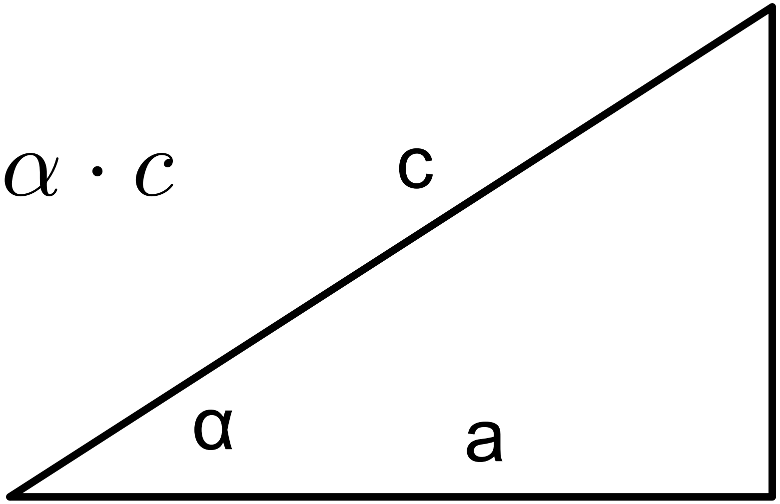
How much do the axioms Capture?

For instance Euclidean geometry can construct a pentagon with compss and straightedge, but not a heptagon.

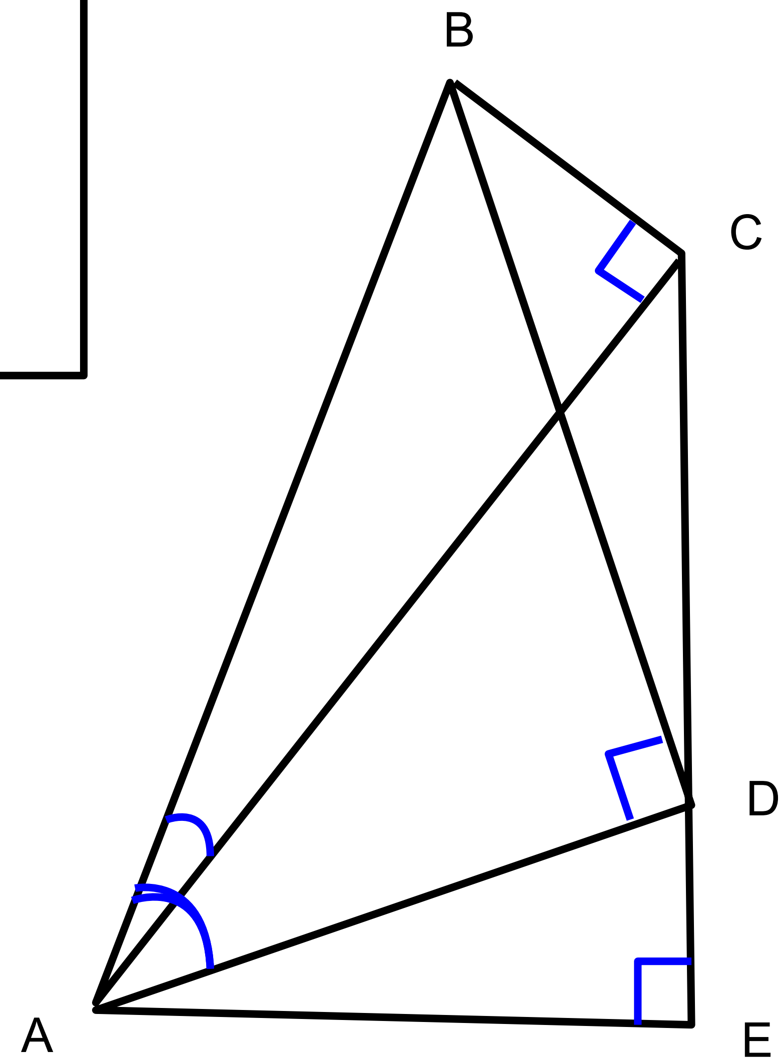
The plan Hilbert used was see what operations were necessary for the axioms by setting up an 'algebra of segments'. This is the opposite of supplying a model that the axioms satisfy as when checking consistency.

First a small lemma

$$a = \alpha \cdot c$$



$$\alpha \cdot \beta \cdot c = \beta \cdot \alpha \cdot c$$

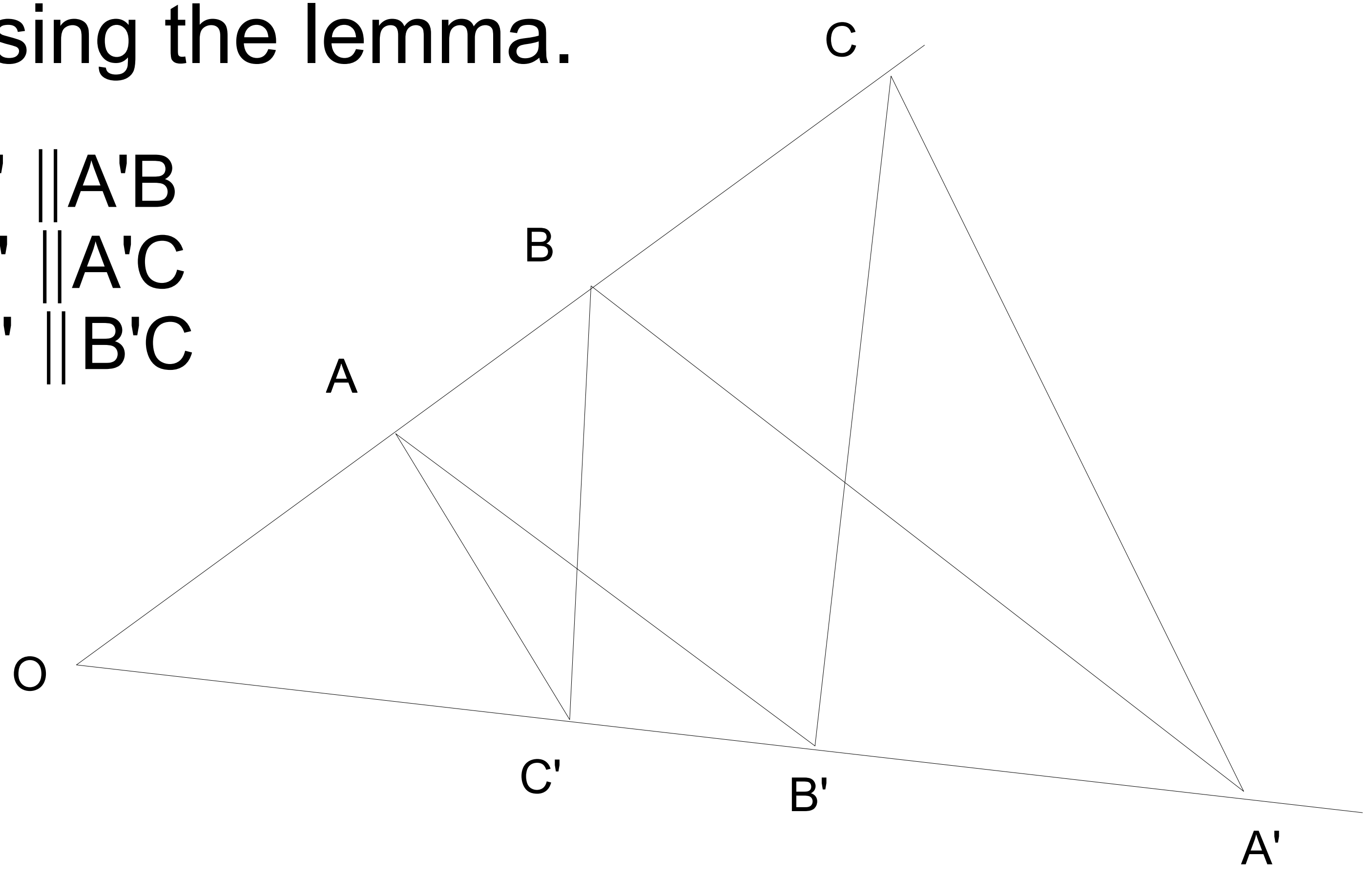


Pascal's Theorem

In fact a very restricted variant of it

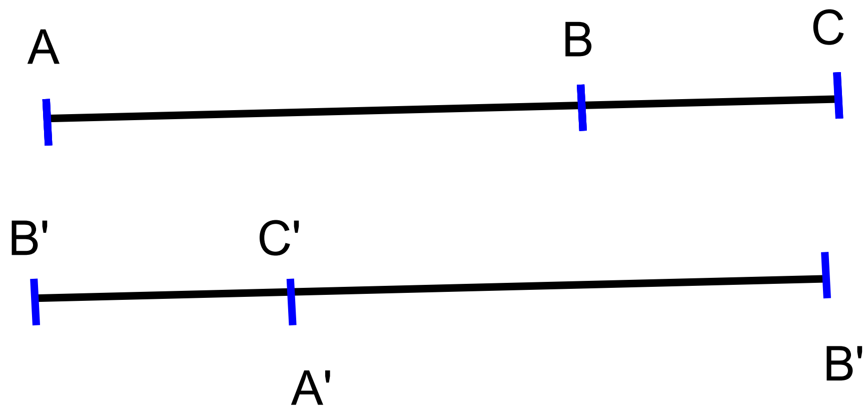
proven by dropping perpendiculars and chasing round using the lemma.

If $\quad AB' \parallel A'B$
and $\quad AC' \parallel A'C$
then $\quad BC' \parallel B'C$

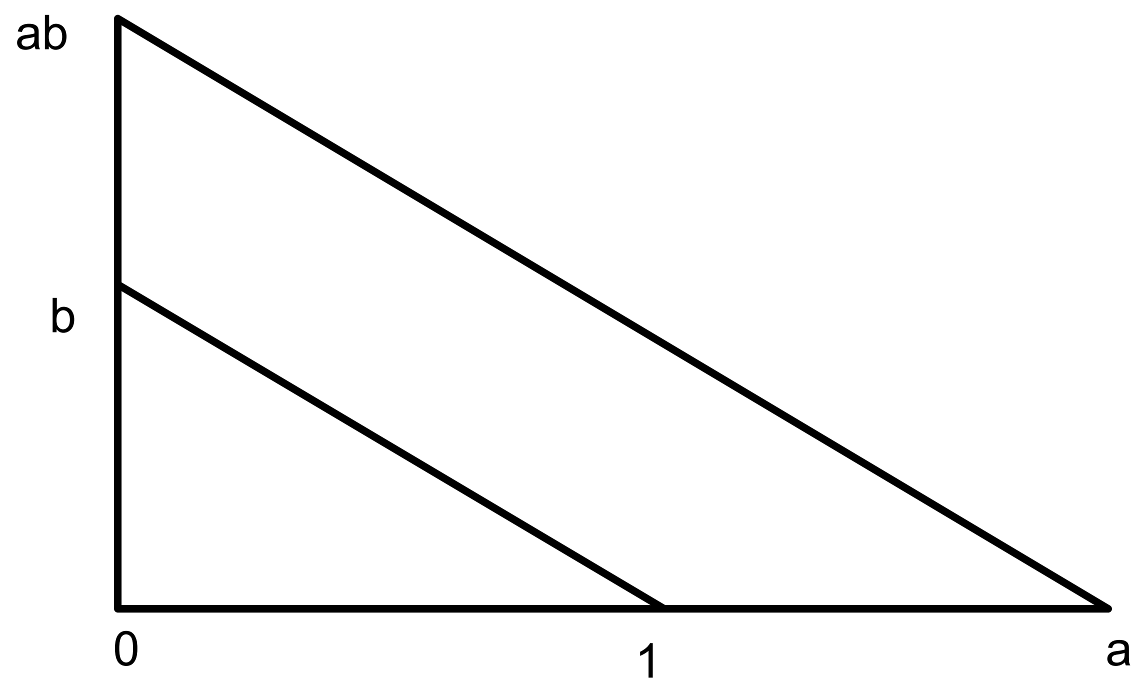


Algebra of segments

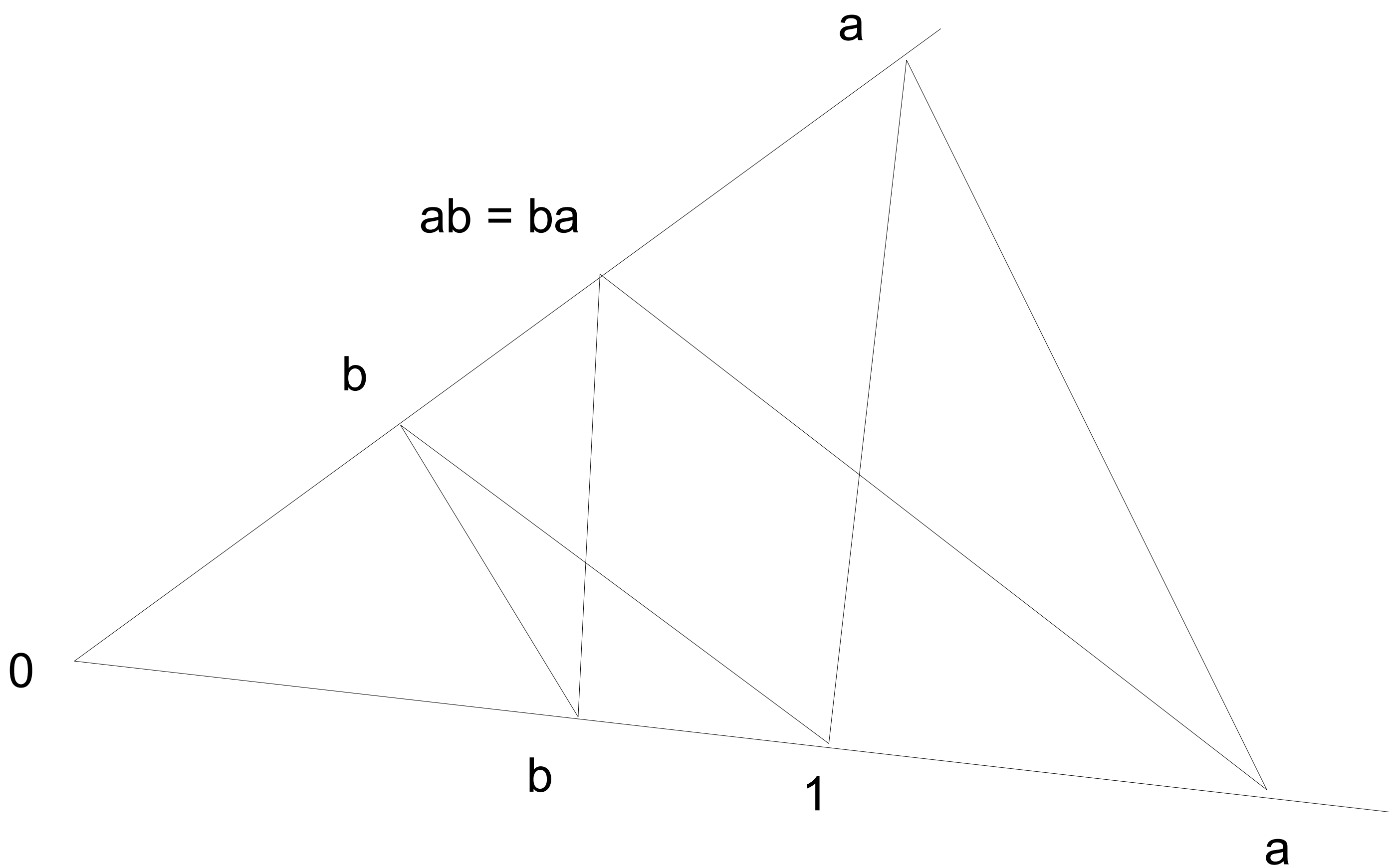
- Addition



- Multiplication



Associativity of Multiplication



Summary of the rest

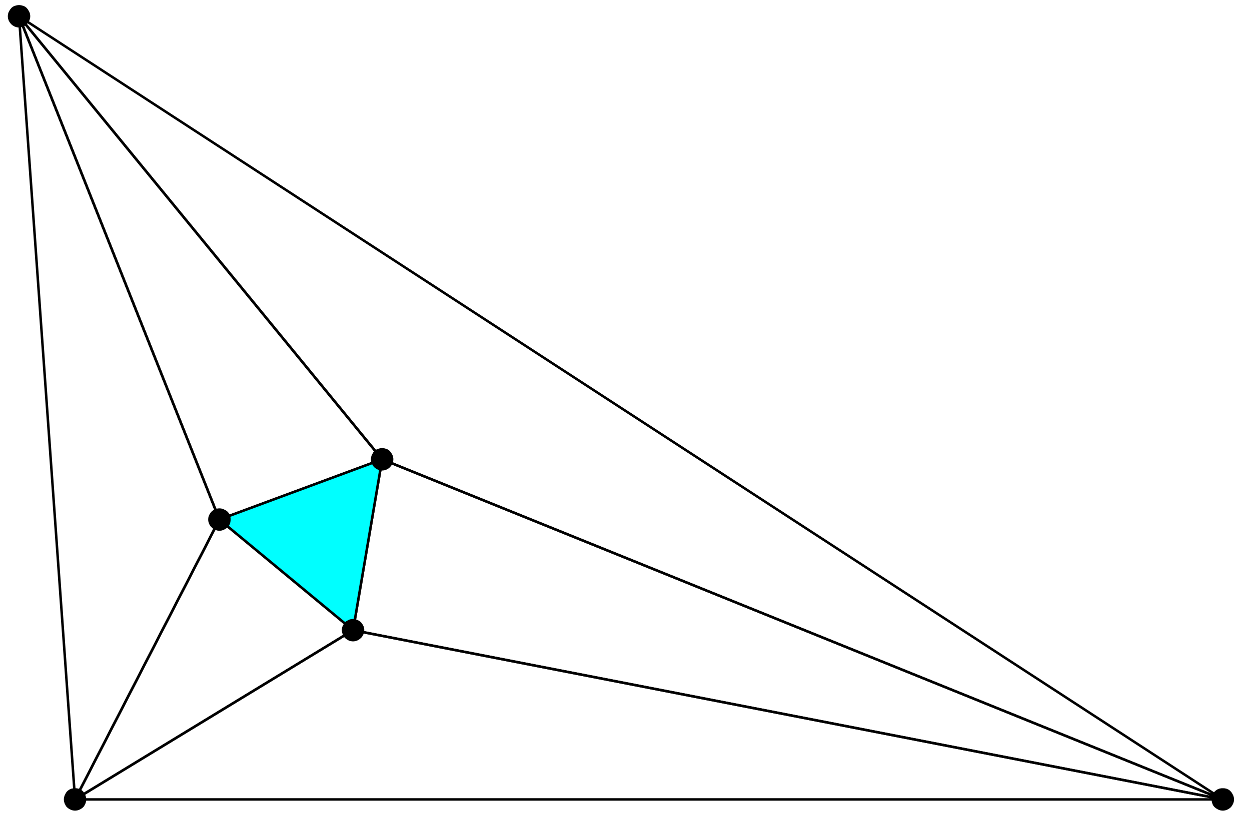
- Show all the axioms of a field are satisfied by the algebra of segments.
- Define area by splitting polygons into congruent triangles. If one is removed the original polygon cannot be made up from the remainder.
- Can then do Pythagoras's Theorem
- So must include $\sqrt{1+x^2}$

Constructible model

- Satifiability is checked by seeing that the axioms are all satisfied in a model with points (x, y, z) where the coordinates are got from 1 and addition, subtraction, multiplication division and $\sqrt{1+x^2}$
- But we also know that all such points can be constructed because those operations are all supported by the algebra of segments.

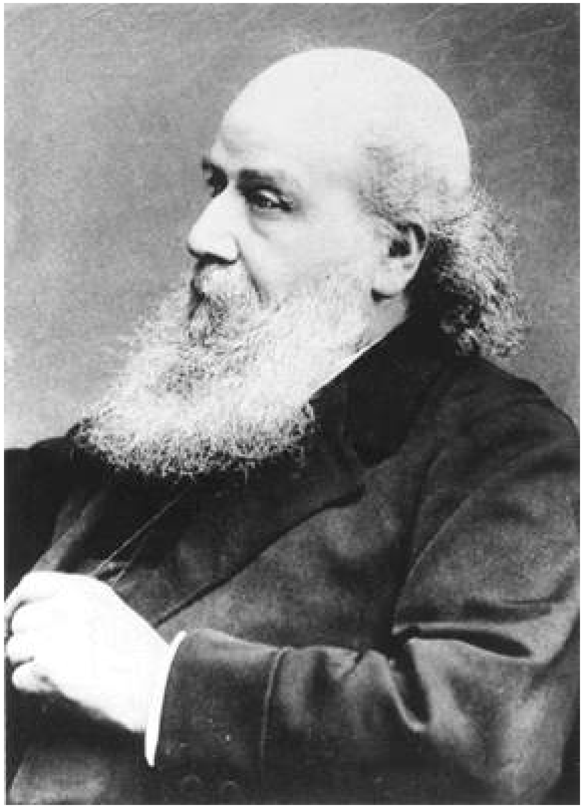
Completeness

- Need competeness for full Euclidean geometry. e.g. for Morley's trisector theorem.



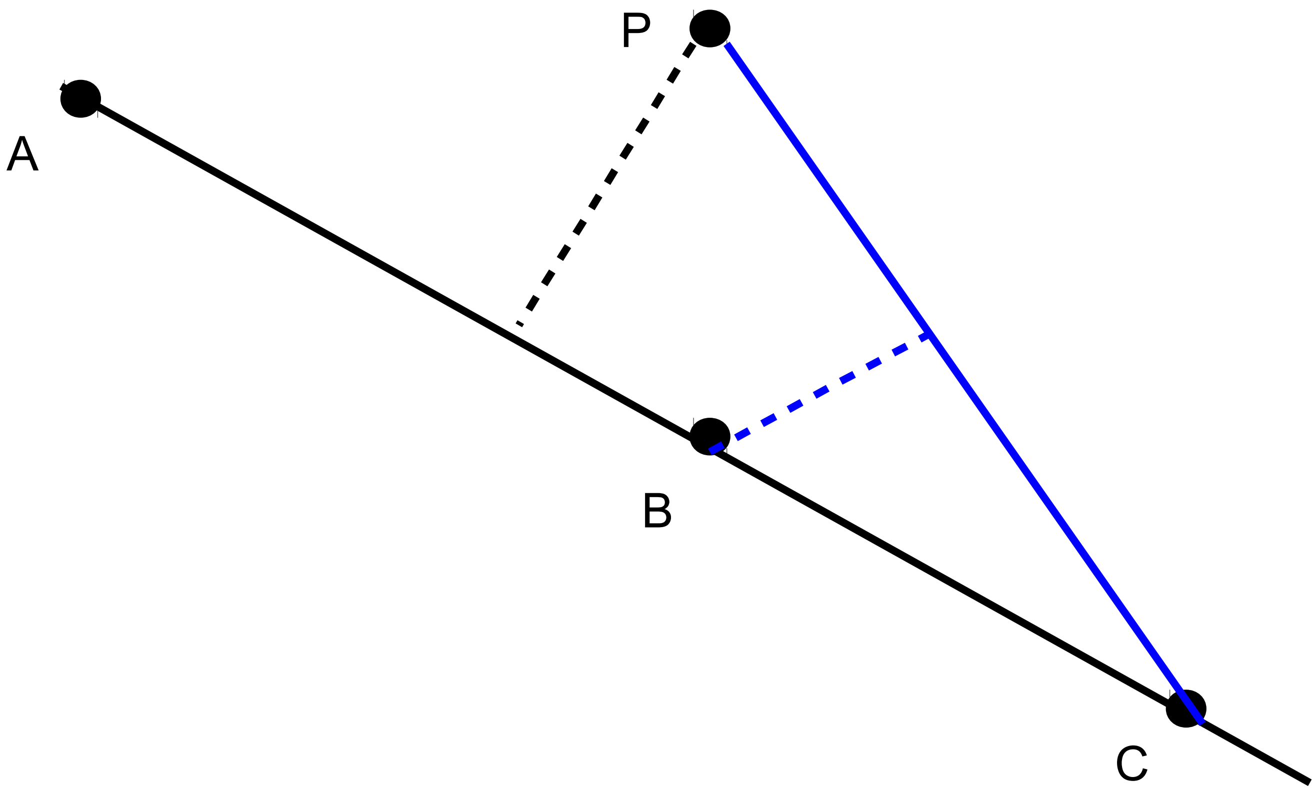
Sylvester-Gallai Theorem

- For a finite number of points on the plane, they are either all collinear or there is a line which contains just two points.

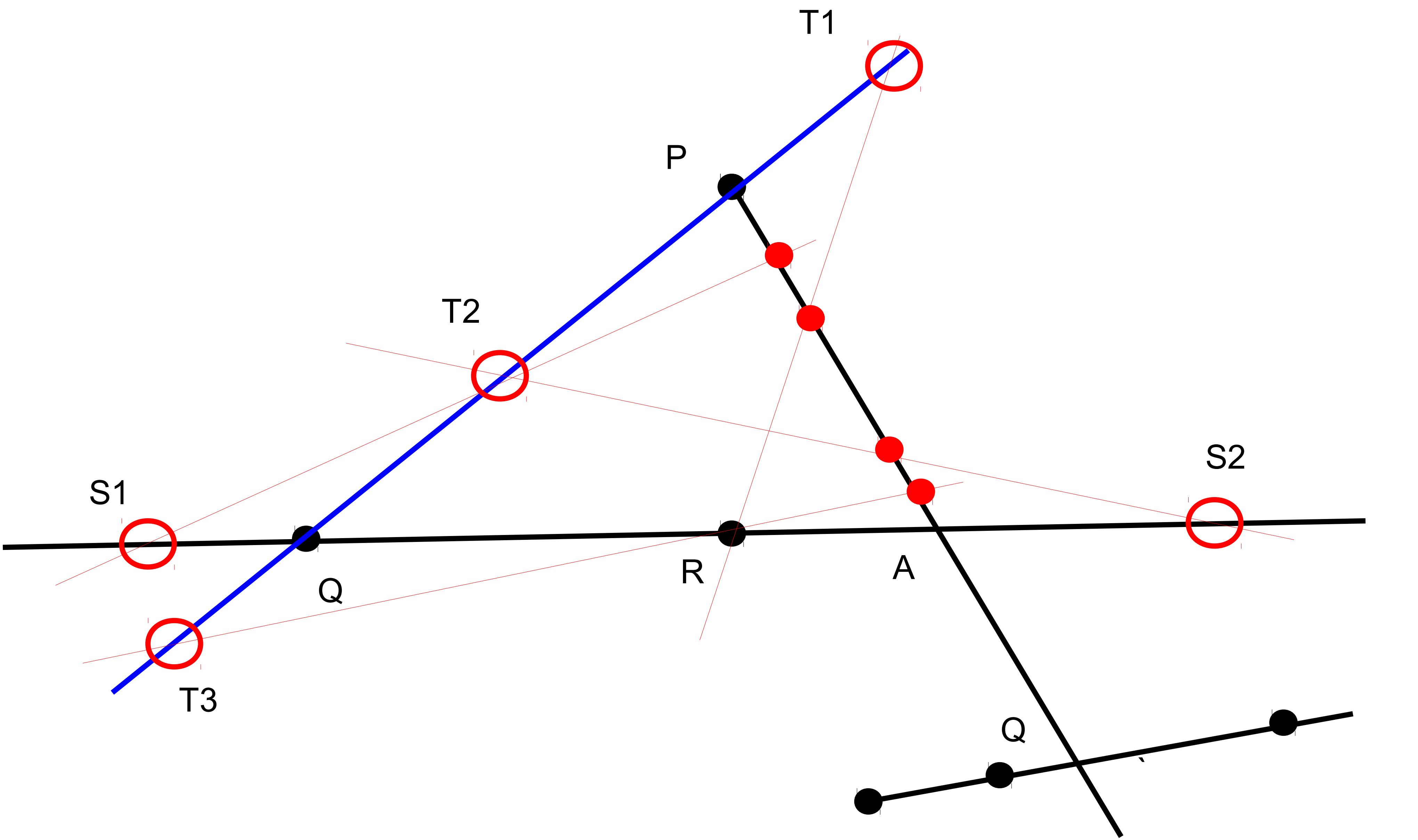


Euclidean proof - Kelly

- Find smallest distance between a point and a line with three or more points.



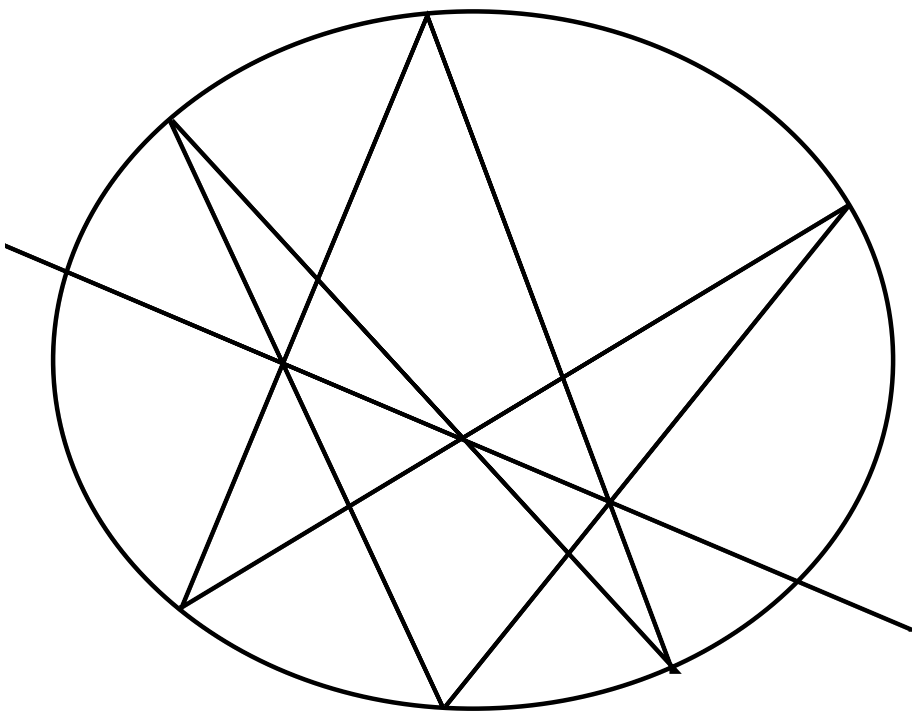
Ordered Geometry - Coxeter

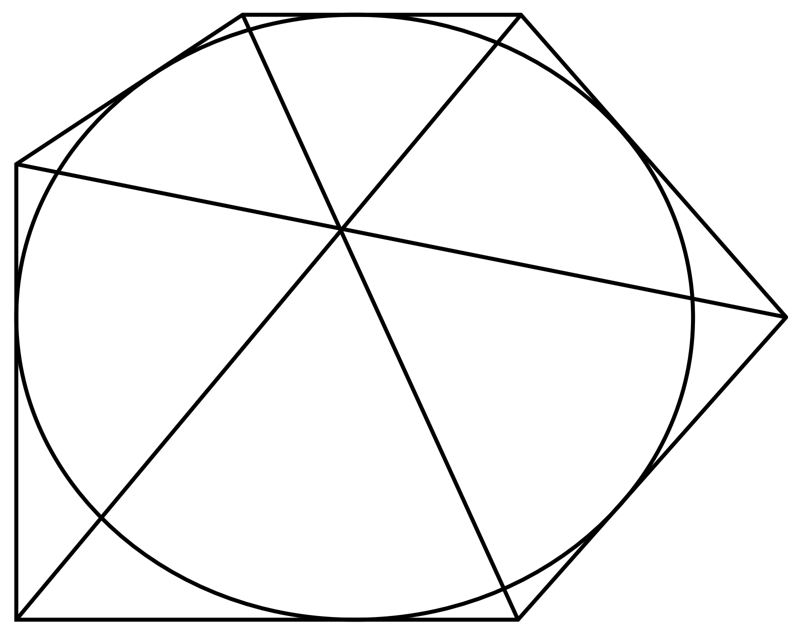


Duality

In projective geometry if lines and points are swapped in the axioms then there is no change – so what's true of one is true of the other.

- Pascal's theorem and Brianchon's Theorem





Duality - Melchior

- Any arrangement of lines has a point where just two lines cross.

  Each face bounded by at least 3 edges

  Each edge bounds two faces, so $3F \leq 2E$

  Assume there must be at least 6V/2 edges

  then V-F+E must be at least

  $$V-2V+3V = 2V$$

  Should be 1 in projective geometry, 2 if Euclidean