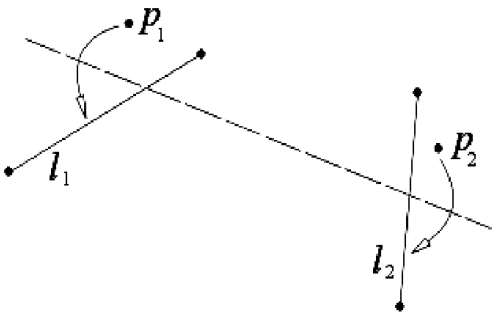
Mathematics of Paper Folding

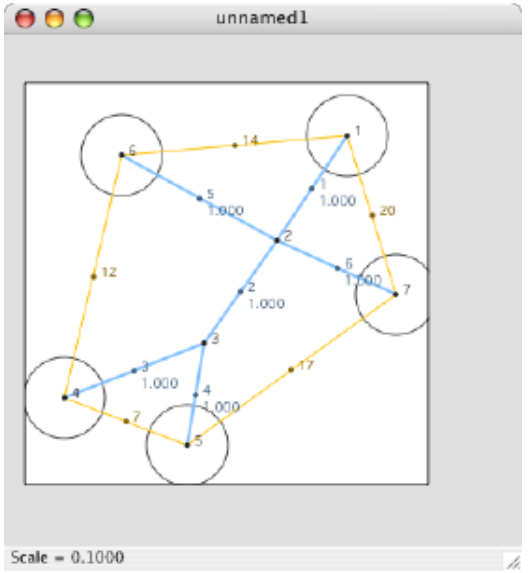
Or mathematics with paper but no pen

David McQuillan

A survey

- The theoretical bits
- Computer support
- Real world uses



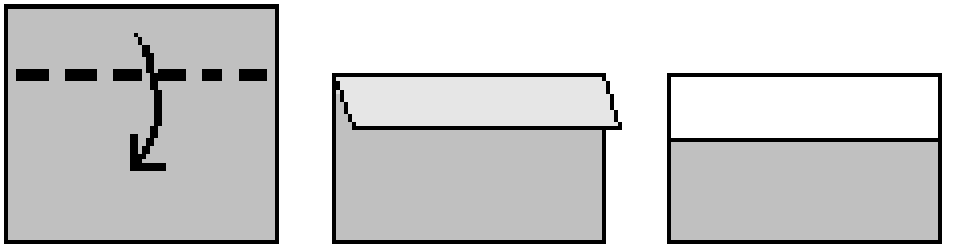


A bit of History

- Origami round the world – Japan, China, Spain, Germany. But an undeveloped hobby or art
- Modern origami dates from 1954 with the publication by Akira Yoshizawa of a diagramming system.
- Mathematical study dates from about 1990 with some earlier bits
- Problems of cutting and folding polyhedrons date to Albrecht Durer in 1525.

The basic language of origami

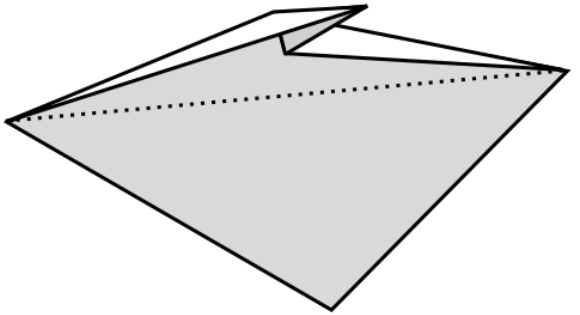
- Valley fold



- Mountain fold

- Compound folds

  Squash, petal, rabbit ear etc.



- Bases – Kite, Fish, Bird, Frog, Blintz etc.

Huzita Hatori Axioms

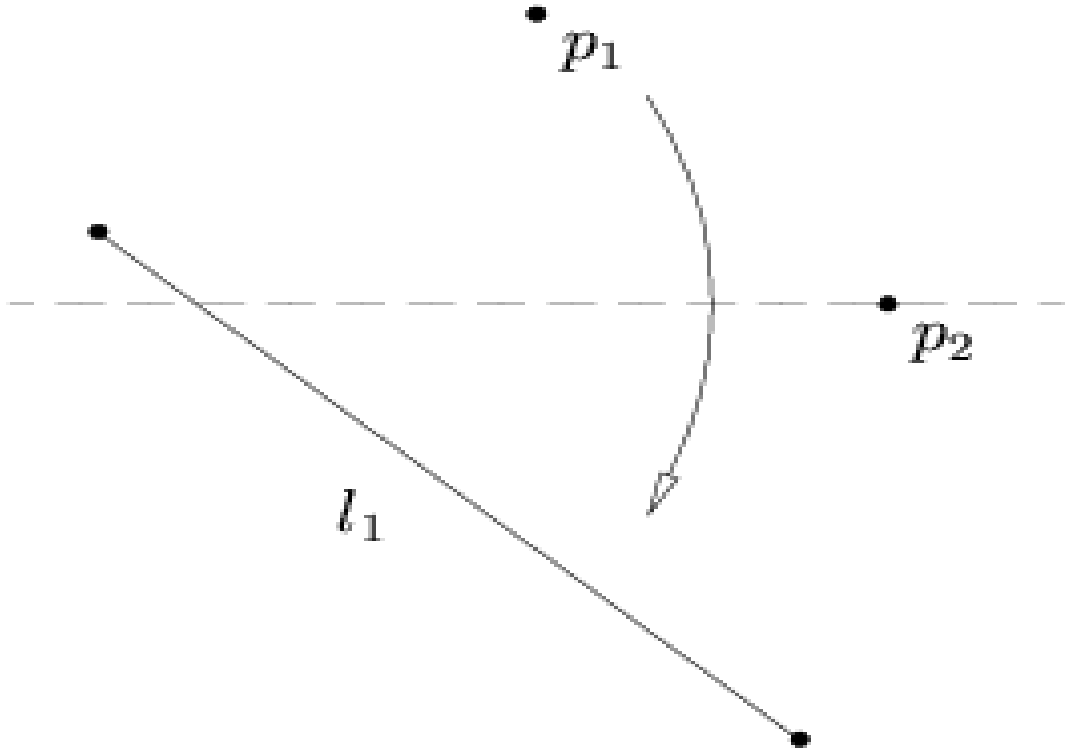
One fold at a time using existing points and lines. The constructable points are where any constructable lines cross and each fold gives a new line.

- First 5 give Euclid plane geometry
- Last 2 enable the the extra constructions allowed by using a marked ruler, the neusis constructions.

The first 5

- Given two point fold a line through them
- Given two points fold a perpendicular bisector
- Given two lines fold an angle bisector
- Given a point and line fold the perpendicular
- Given two points and a line fold a line that goes through one point and places the other point on the line. This is equivalent to finding the intersection of a line and a circle

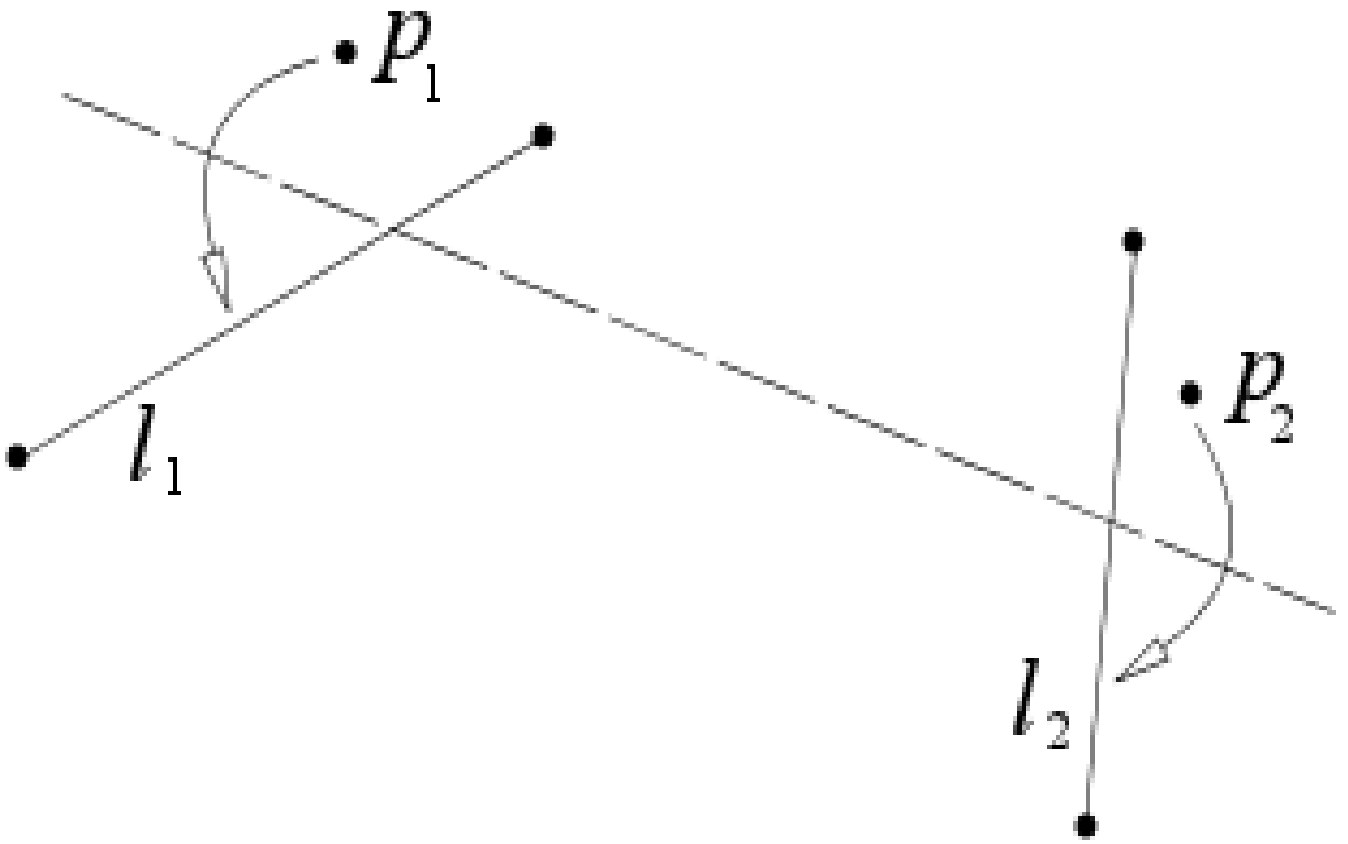
Axiom 5



In effect find where circle centre p_2 and radius p_2p_1 cuts the line l_1 and then bisect the line that connects it to p_1

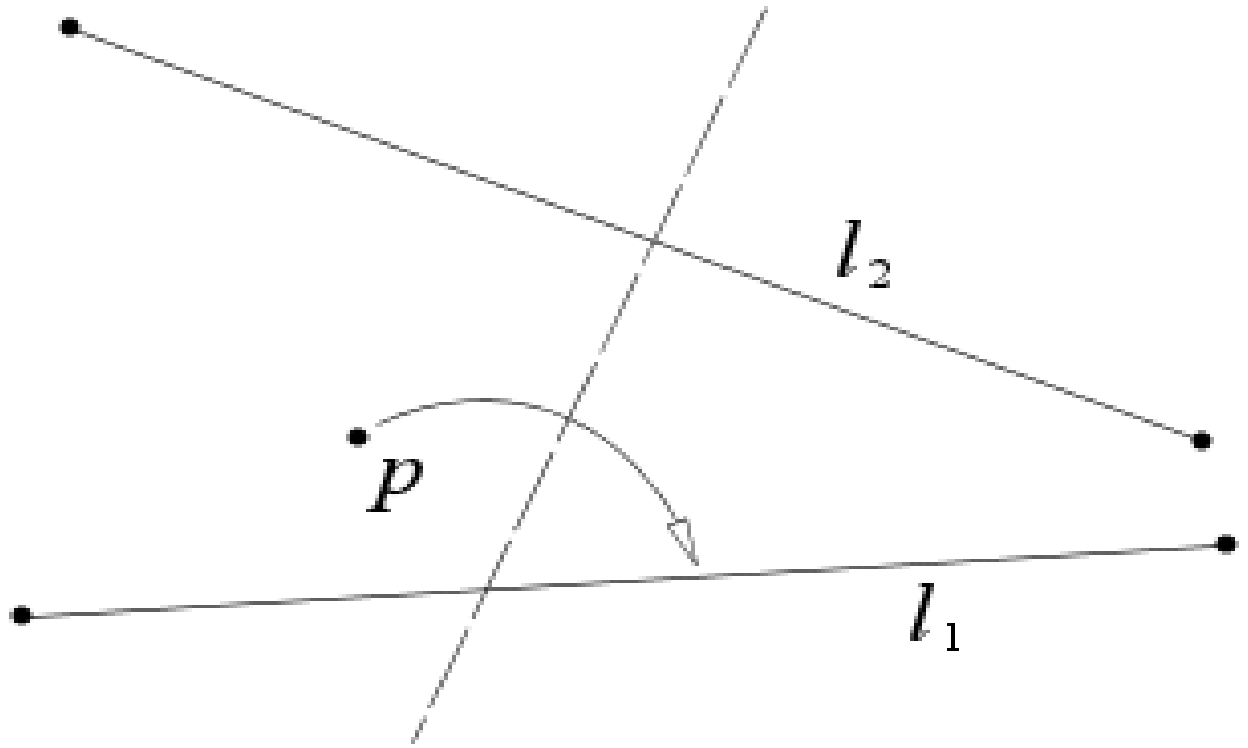
Humiaki Huzita's Axiom 6

- Given points $p_1 p_2$ and lines $l_1 l_2$ there is a fold that places p_1 on l_1 and p_2 on l_2



Koshiro Hatori's Axiom 7

- Given point *p* and lines l_1 and l_2 there is a fold that places *p* on l_1 and is perpendicular to l_2



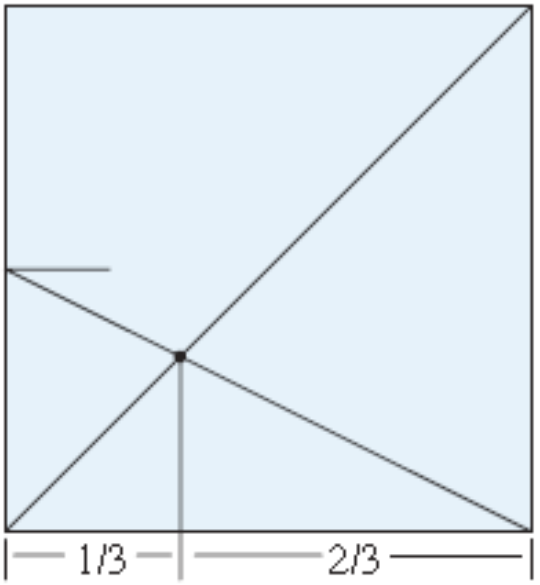
What points can be constructed?

- Robert Lang has shown the 7 axioms give all the points that can be constructed using single folds.
- Axioms 1, 2 and 4 gives a rational field.
- Axiom 3, the angle bsector, gives the Pythagorean field including square roots
- Axiom 5 gives the Euclidean field closed under the square root operation.
- Axioms 6 and 7 add cube roots
- Some redundancy in the 'axioms'

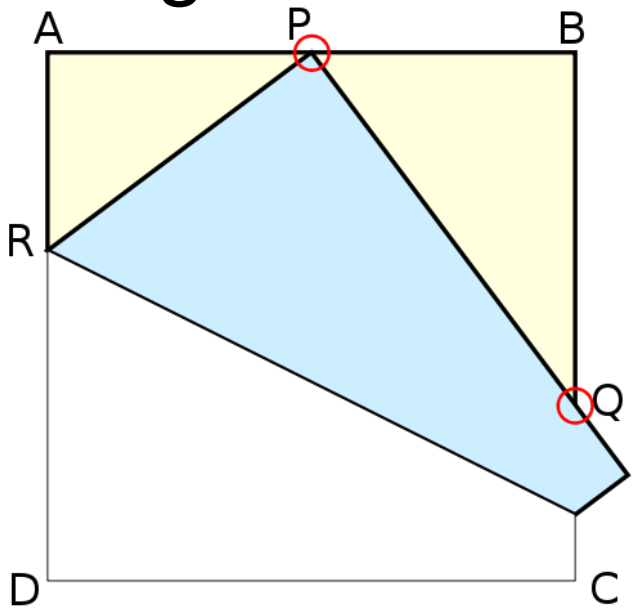
Rational division

- Many ways to divide the side of a square in a rational fraction

Crossed diagonals



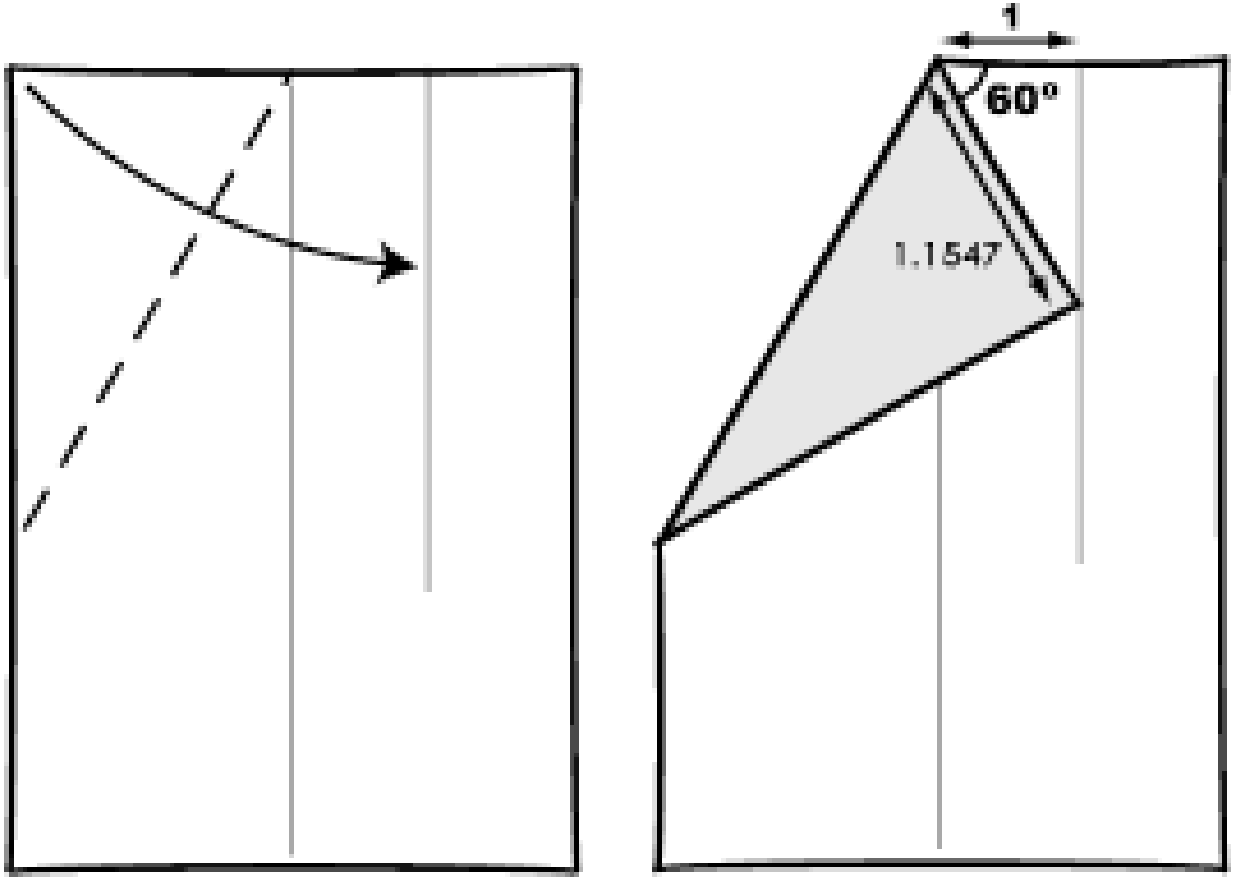
Haga's theorem



Also Fujimoto's and Noma's constructions

Square roots

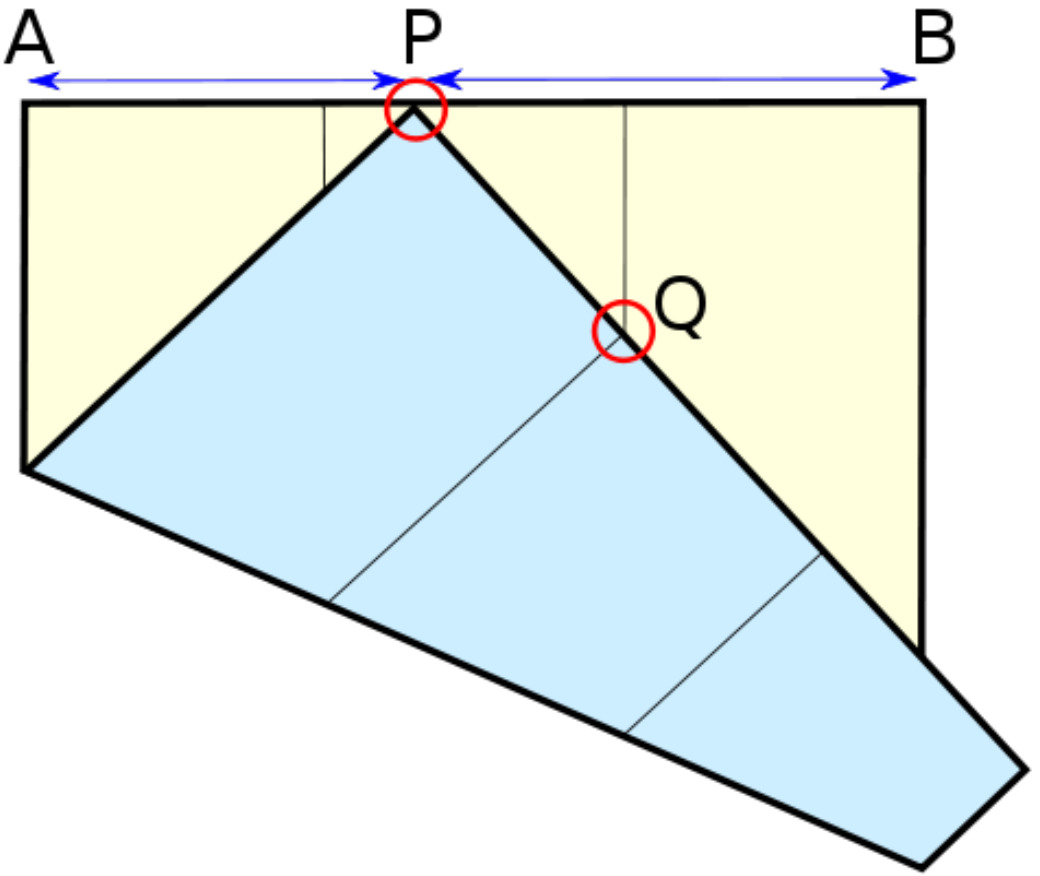
Axiom 5 allows square roots to be constructed



The Delian Problem

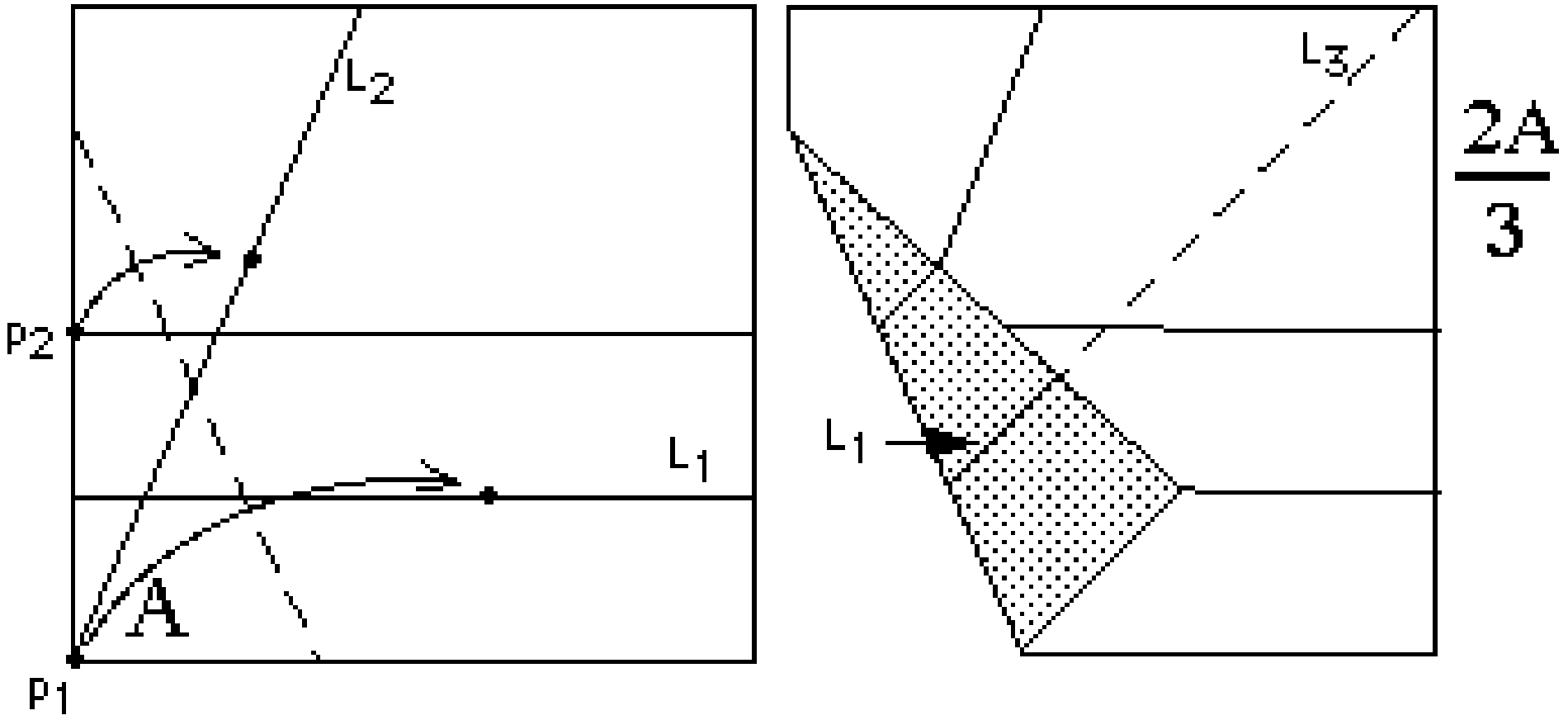
PB / PA = the cube root of 2, using axiom 6

(Peter Messer)



Trisecting an angle

Another use of axiom 6 (Tsune Abe)



Regular polygons

- Euclidean geometry can construct polygons with sides

  $2^a p_1 p_2 p_3 ...$

  where p_i are different Fermat primes $2^{2^n}+1$

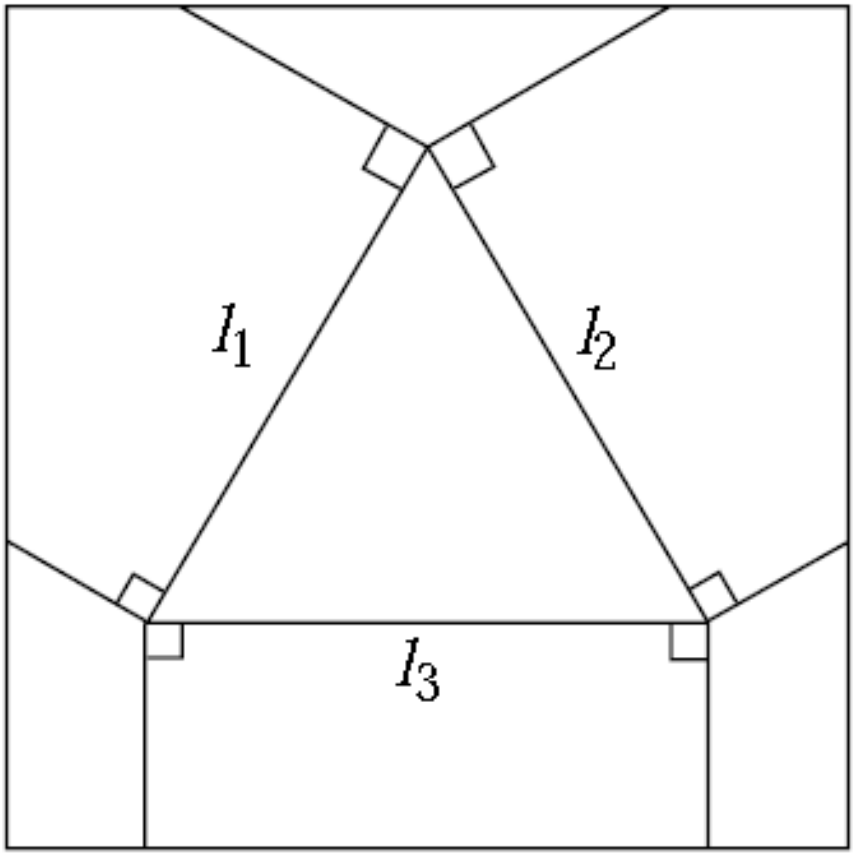
- Origami can construct

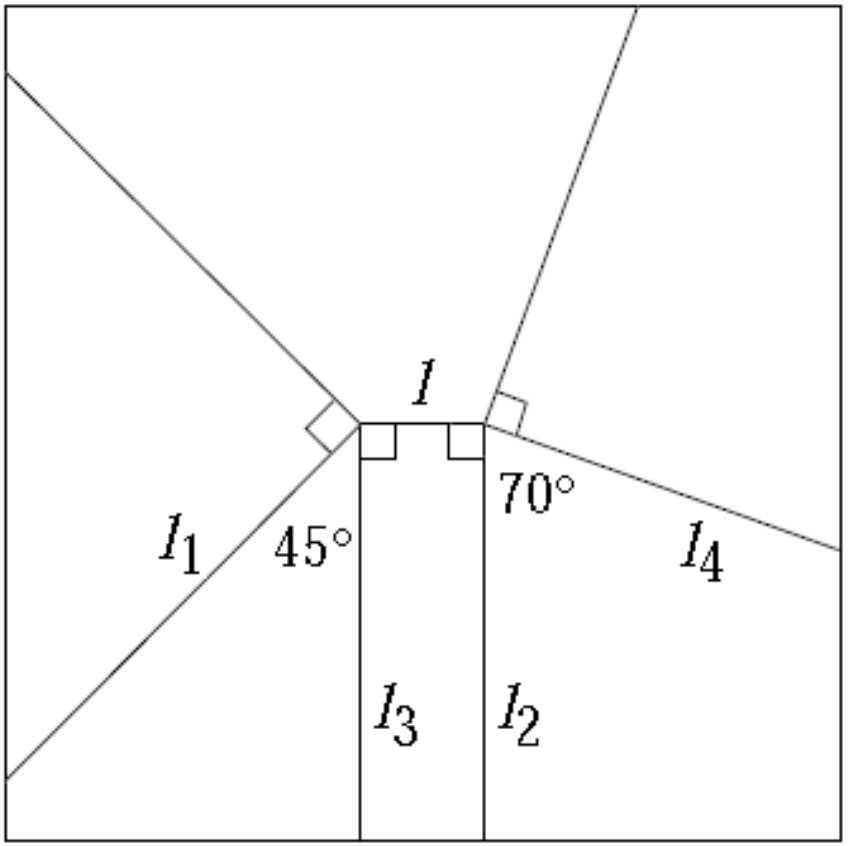
  $2^a 3^b p_1 p_2 p_3 ...$

  where p_i are different Pierpoint primes $2^c 3^d+1$

Flat folding

- Can these be folded flat?





Necessary conditions

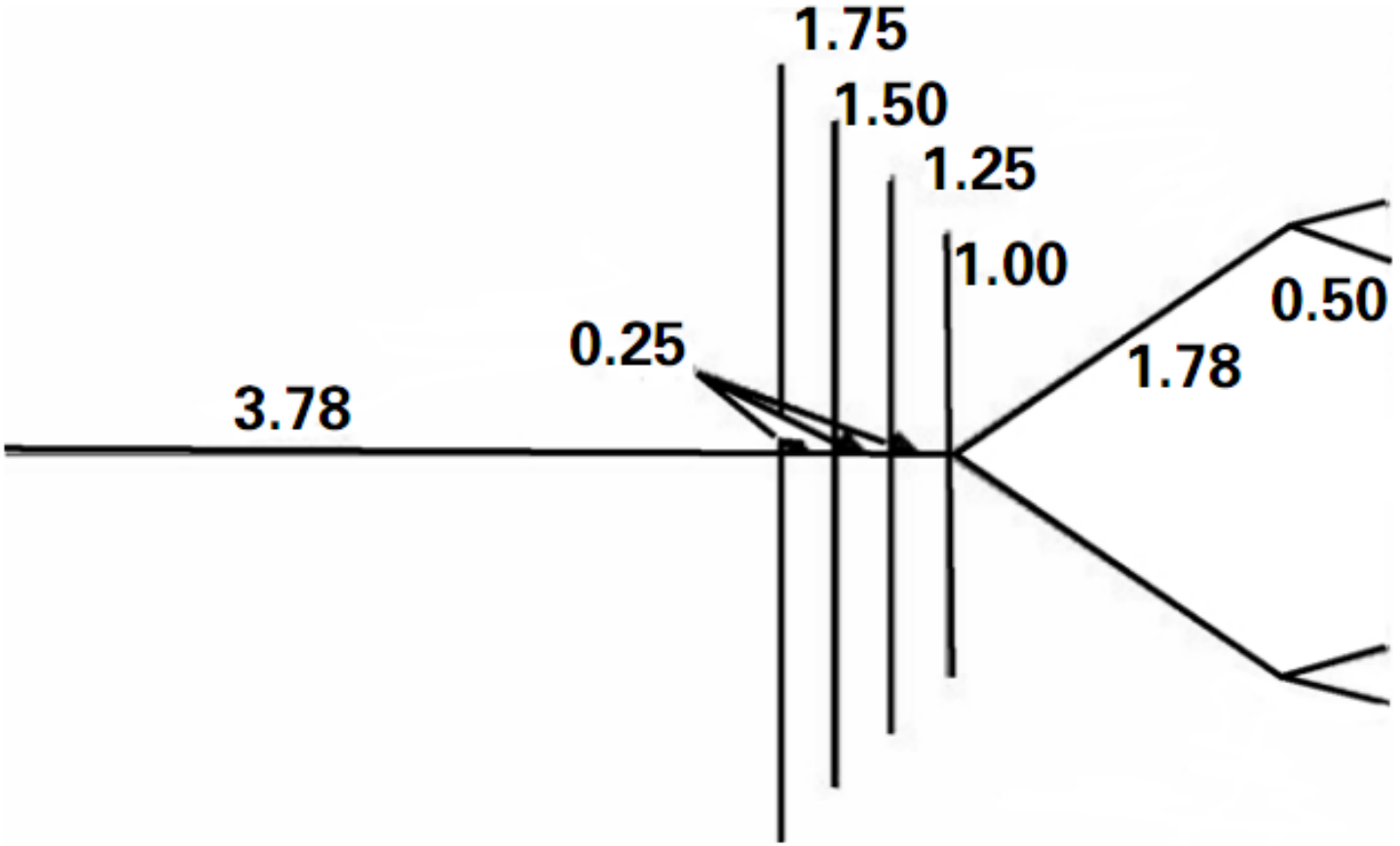
- The crease pattern is two colourable
- The number of valley and mountain folds differs by two (Maekawa's Theorem)
- The sum of the alternate angles round a vertex adds up to 180 degrees (Kawasaki's theorem)
- And of course we need that the sheets don't penetrate a fold!
- In general an NP complete problem

Computer support

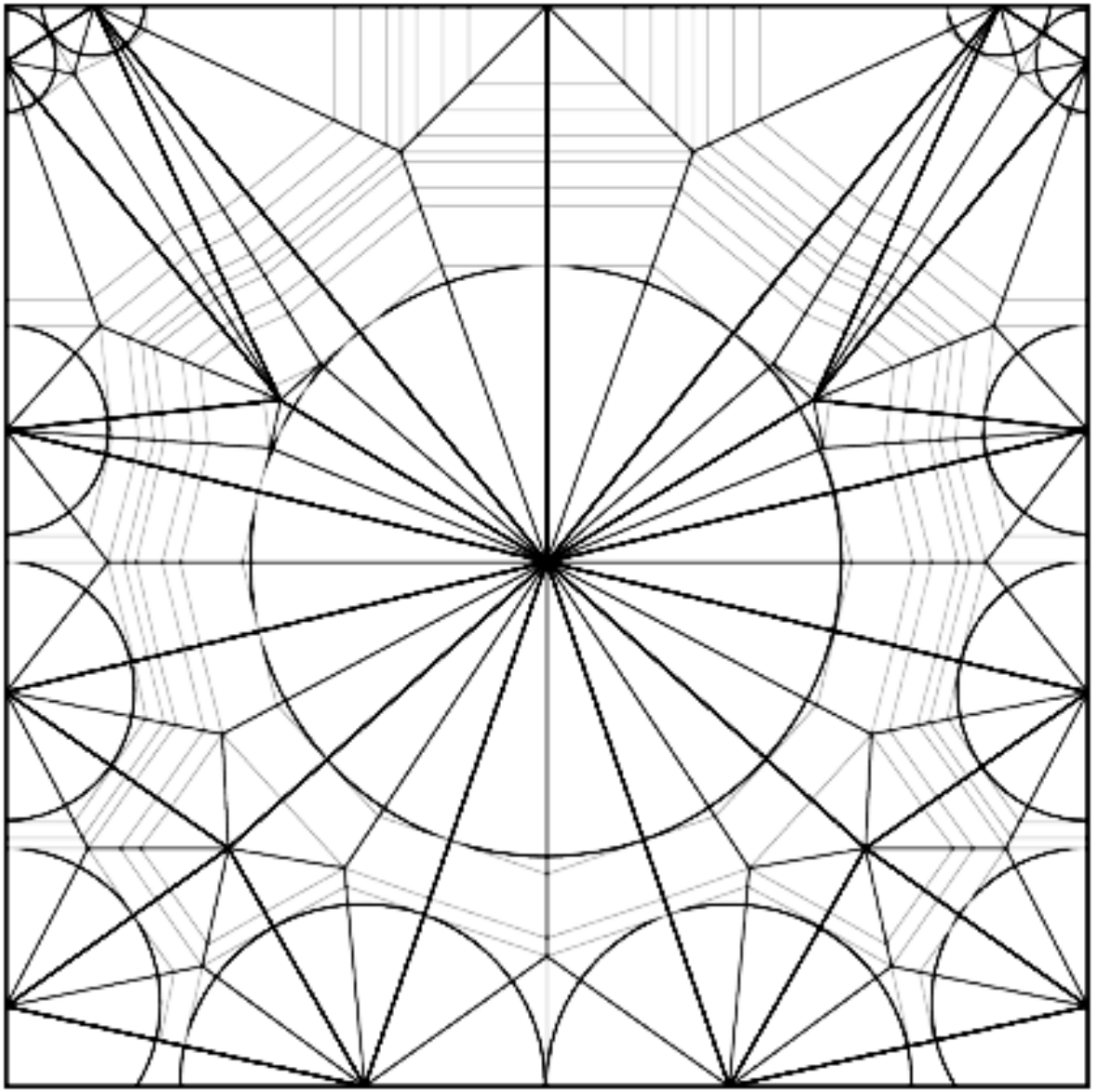
- TreeMaker makes a base with a given skeleton and flaps of a given relative size by packing circles.
- ReferenceFinder which a simple way to get a given point or line using folding
- ORIPA – Origami pattern editor, will try to fold a crease pattern
- OrigamiSD – Virtual origami folder

TreeMaker

Robert Lang making a scorpion

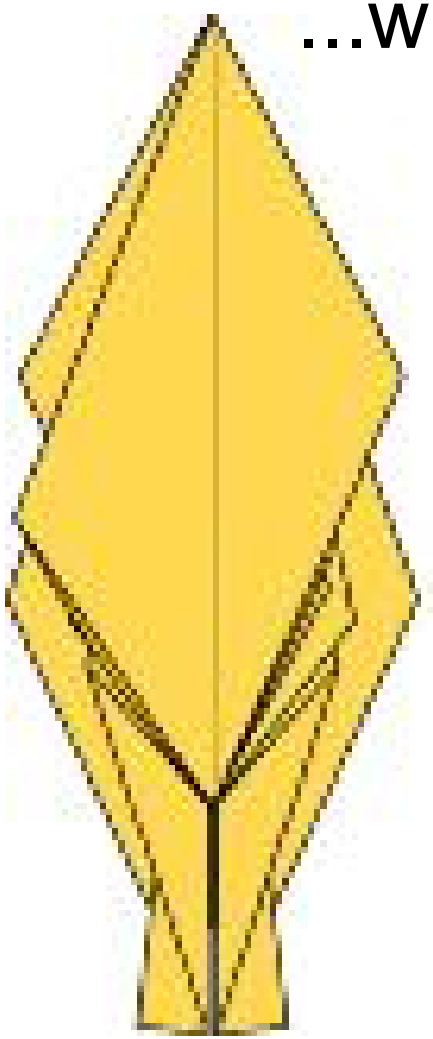


TreeMaker crease patttern



The scorpion

...with a bit of artistry from Robert Lang



But is it useful?

Some application of straightforward origami, but most application use '**Rigid origam**i' which is a variant of the study of mechanical linkages.

- Flat rigid sheets with hinges
- The sizes can be calculated rather than folded
- Not necessarily flat or square

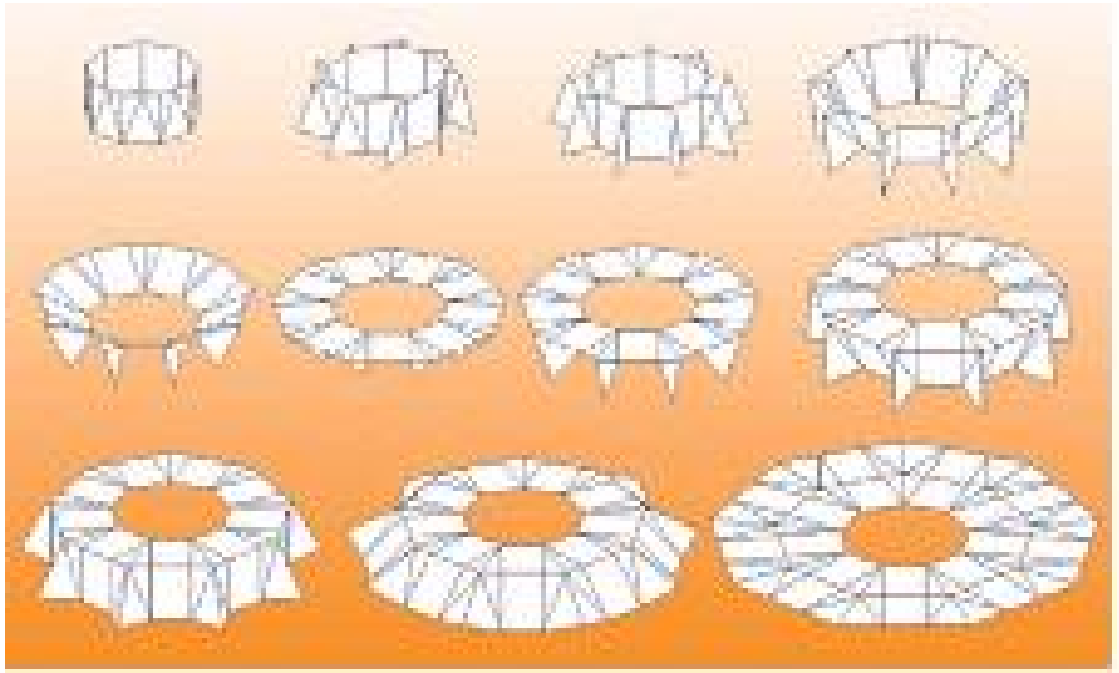
Rigid origami

- Miura map fold.Used to deploy satellite solar panels

- The Bellows theorem, proved in 1997 by Robert Connelly. I. Sabitov, and Anke Walz. The volume of a closed flexible polyhedron is constant under flexing. Causes problems for airbags.

Eyeglass

- A 100m space telescope
- Frensel lens folded to go into space



Solved and unsolved problems

An ancient art with a modern take

- Napkin problem – can one make the perimeter longer than the original square – yes
- Flatten a convex polyhedron into a connected polygon – unsolved
- Can a paper bag be folded flat using rigid origami – yes
- ...etc