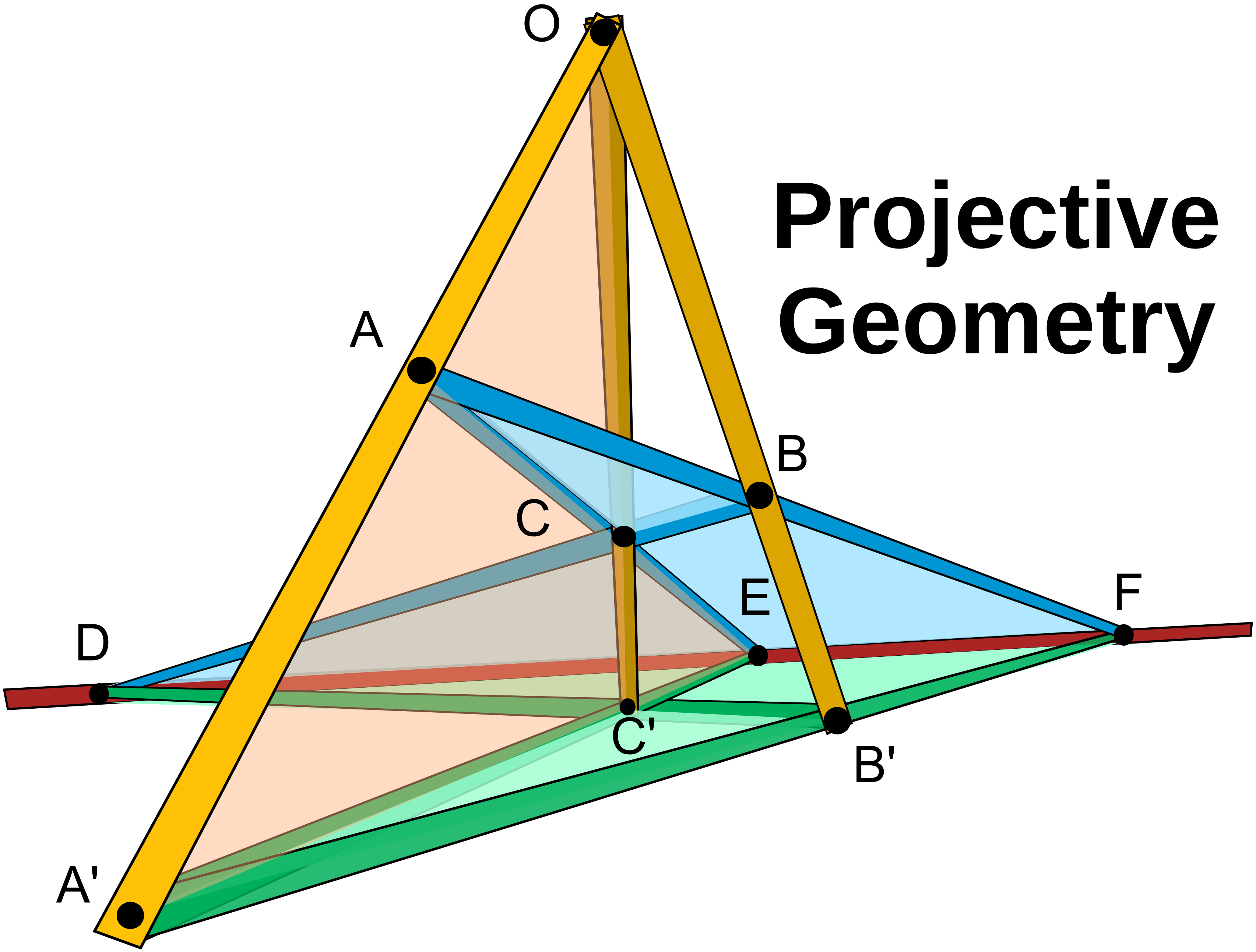
Projective Geometry



Lines and points like Euclid but...

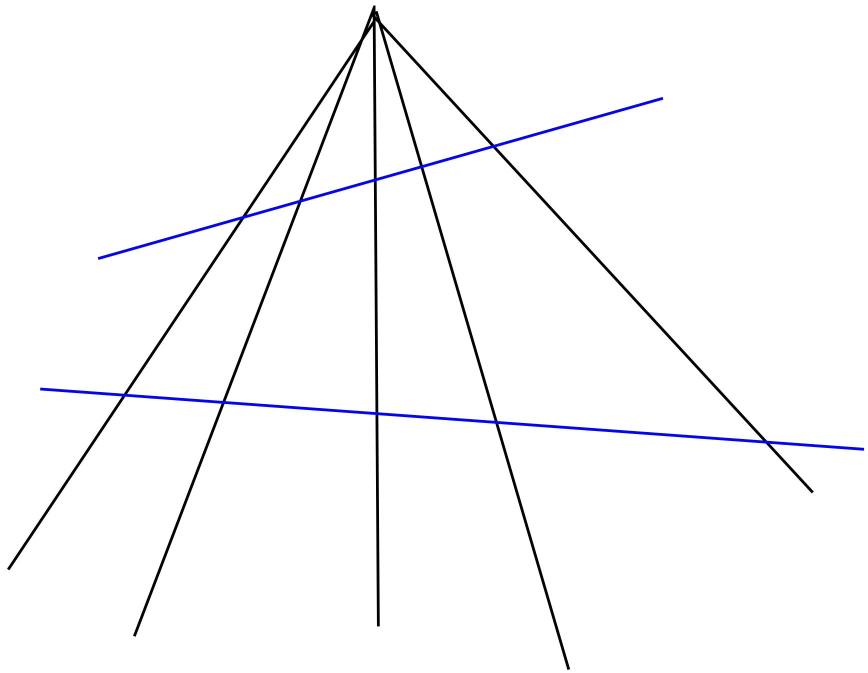
- Nothing about angles
- Nothing about similarity
- Nothing about distances
- Nothing about parallel lines
- Nothing about areas
- Nothing about circles

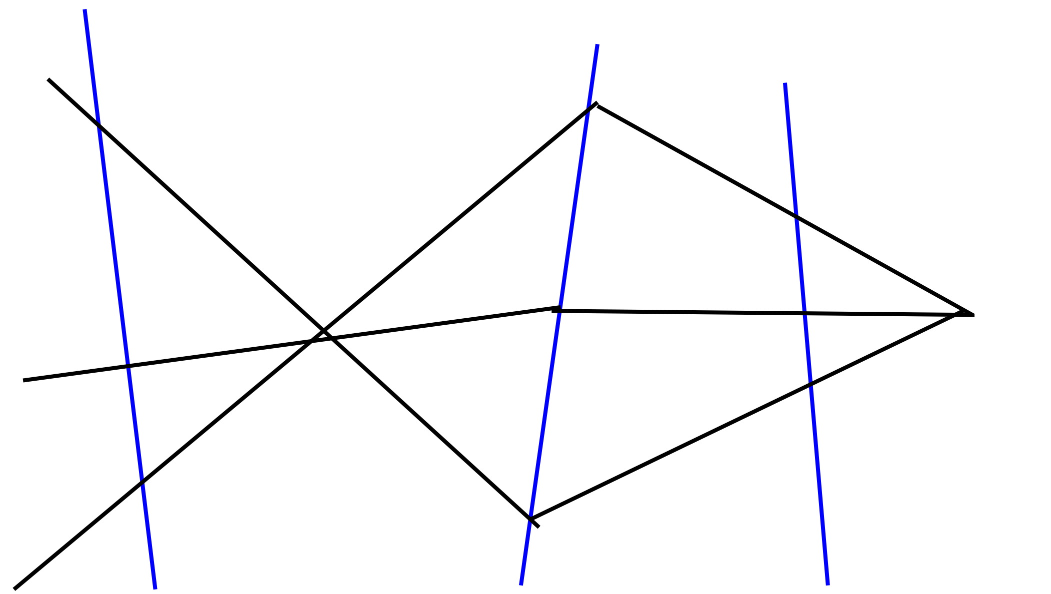
That's most of Euclid gone -what could be easier?

Perspectivity and Projectivity

Perspectivity is a single step

Projectivity is a sequence, two are the same if they have the same effect.

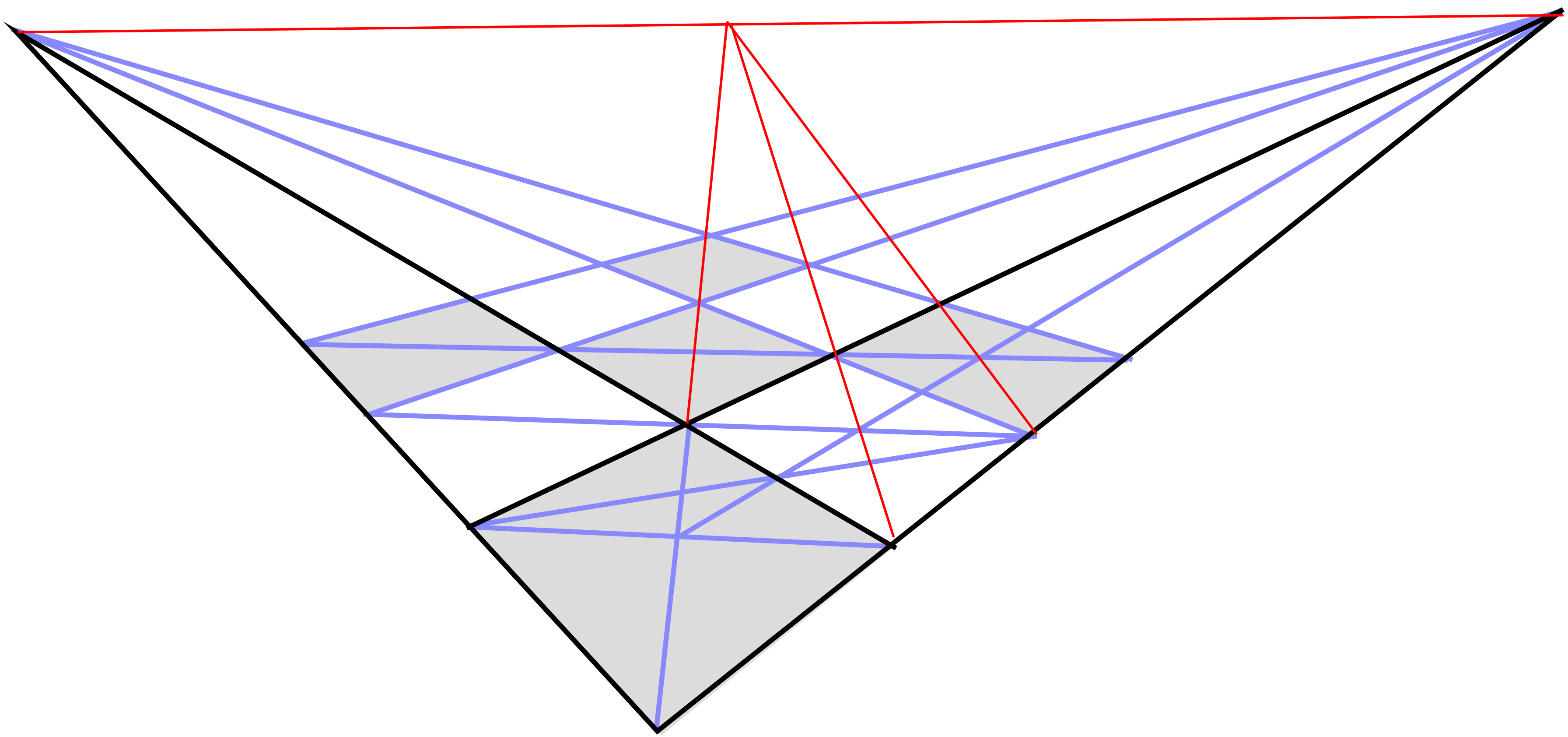




Perspective drawing

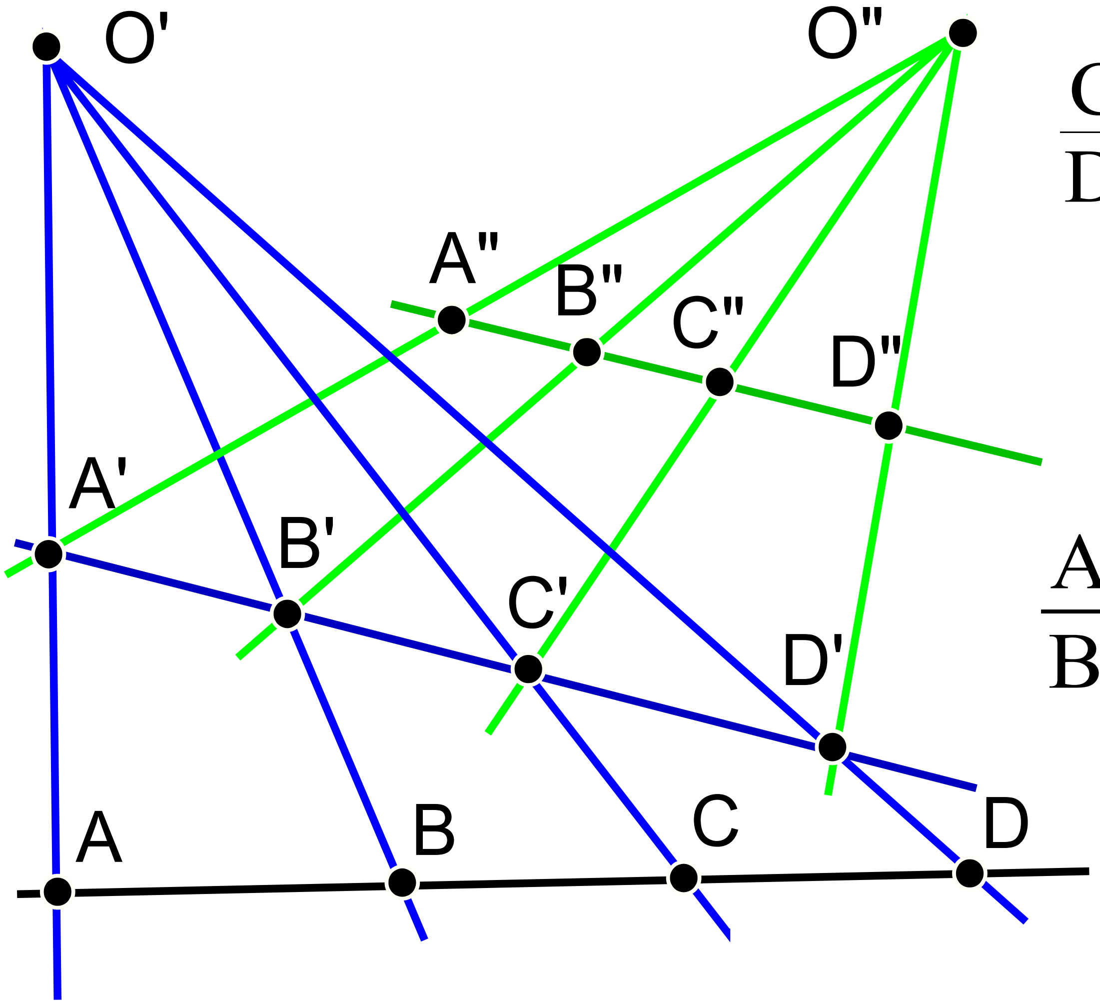
Line of points at infinity

- where parallel lines meet



Cross ratio

A version discovered by Pappus of Alexandria ~ 340AD.



$$\frac{CA/CB}{DA/DB} = \frac{AC \cdot BD}{BC \cdot AD}$$

$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{A'C' \cdot B'D'}{B'C' \cdot A'D'}$$

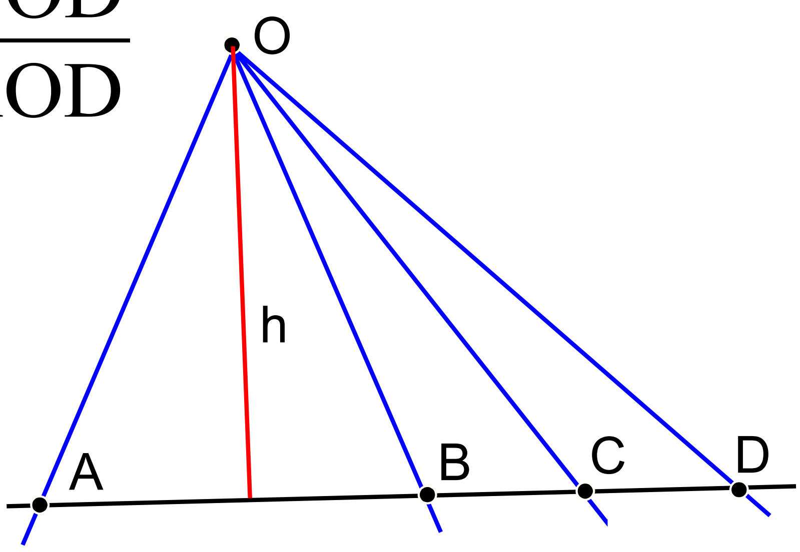
Cross Ratio - proof

$$\text{Area AOC} = \frac{h \cdot AC}{2} = \frac{1}{2} OA \cdot OC \cdot \sin AOC$$

After a bit of cancelling we get

$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{\sin AOC \cdot \sin BOD}{\sin BOC \cdot \sin AOD}$$

So the cross ratio can be associated with the angles at O instead of the points on the line



The Fundamental Theorem

A projectivity is determined when the mapping of three points on one line to three on another is specified.

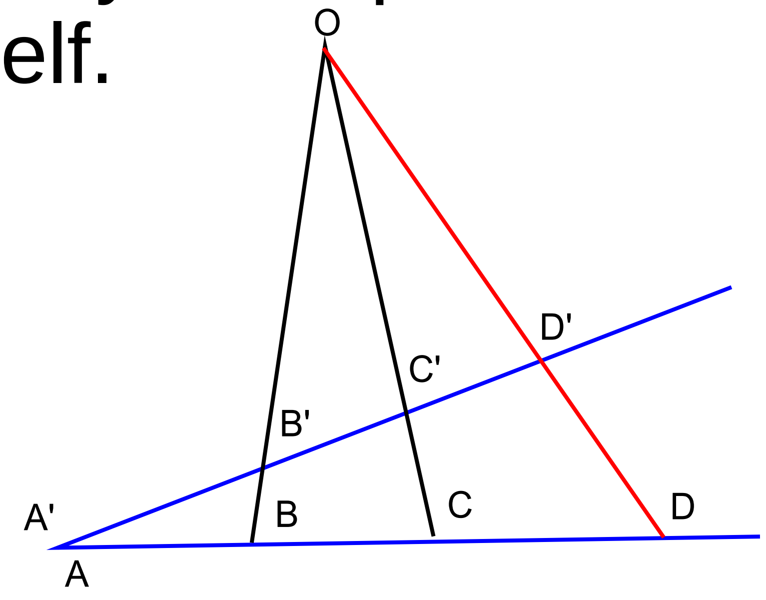
Corollary

A projectivity is a perspectivity if the point where two lines cross maps to itself.

$$ABCD \barwedge A'B'C'D'$$

$$O = BB' \cdot CC'$$

$$BCD \overset{O}{\doublebarwedge} B'C'D'$$



Pappus's hexagon theorem

Derived using Menelaus' theorem by Pappus

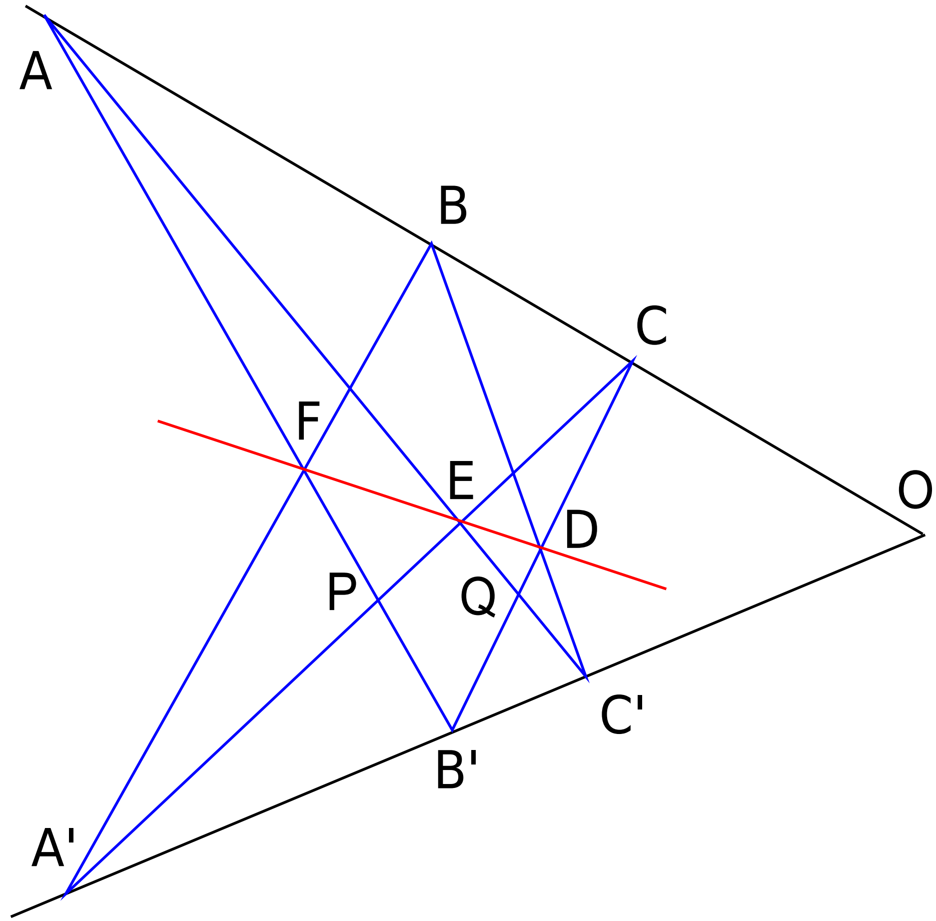
$$AFPB' \overset{A'}{\doublebarwedge} ABCO$$

$$\overset{C'}{\doublebarwedge} QDCB'$$

B' is common

$$AFP \overset{E}{\doublebarwedge} QDC$$

E lies on FD



Later History

- Theory of perspectives for drawings developed in the 15th century
- Desargues Theorem published in 1648 – the picture at the beginning.
- Pascal's theorem in 1640 about a hexagon in a conic
- Main development in the 19th century - duality and axioms

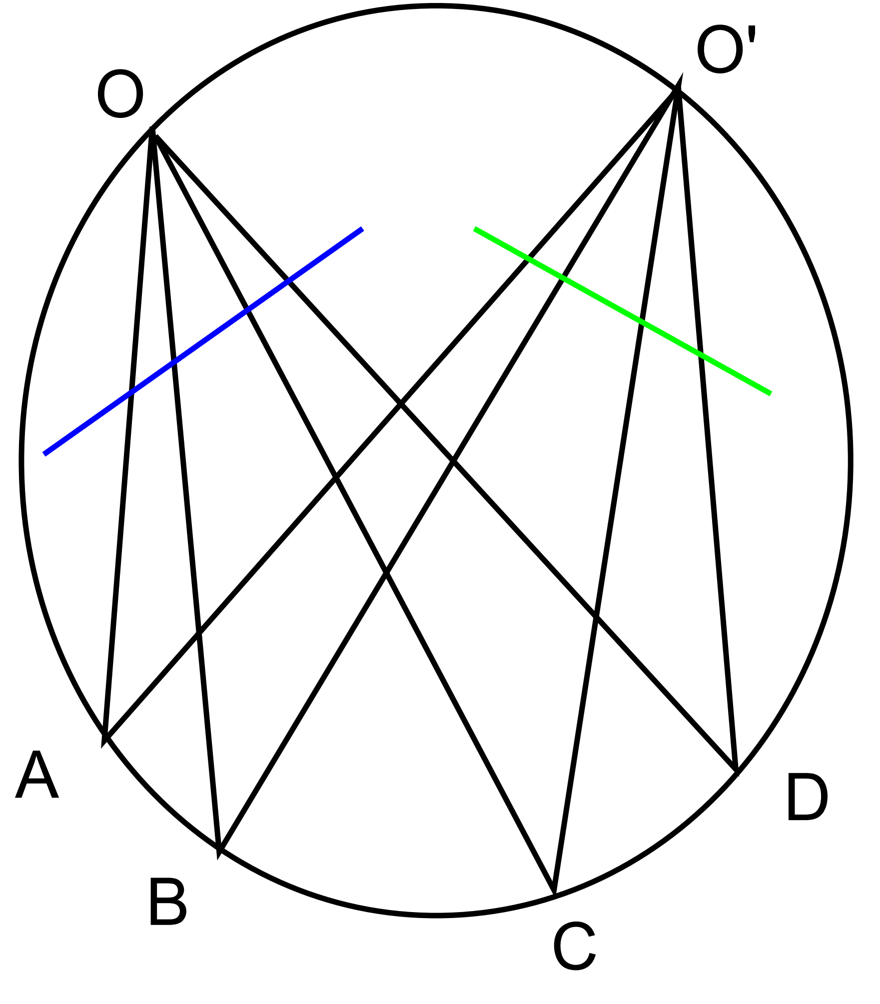
Cross ratio in a Circle

Angles at O and O' are the same

So the cross ratio at O' is the same as at O

Even in perspective

A conic is determined by 5 points



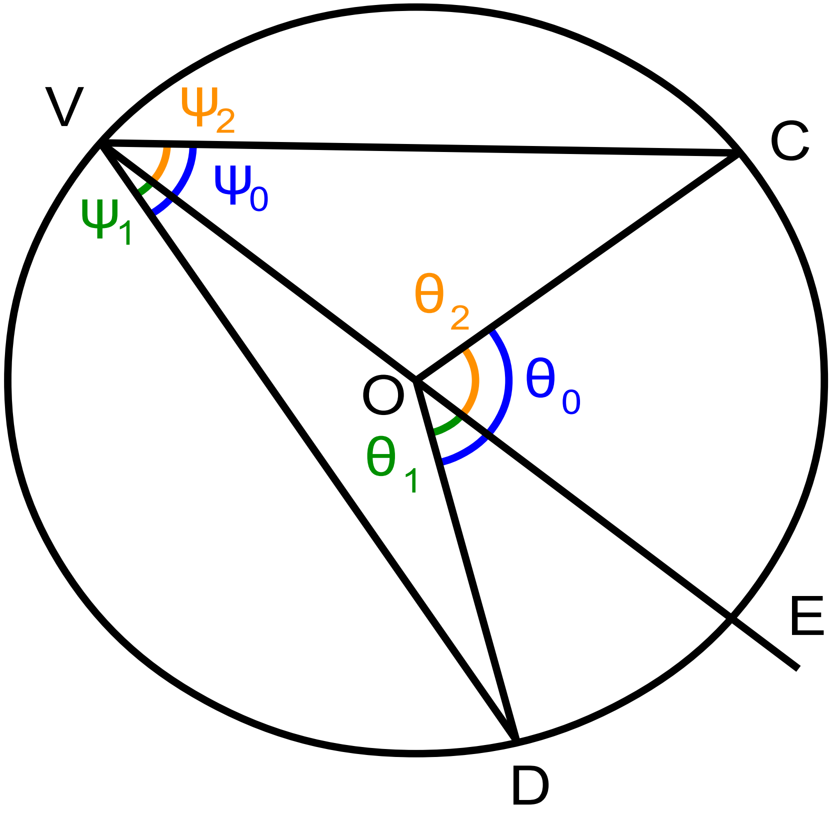
Angle subtended by an arc

Angle at V is half the angle DOC

So it doesn't

depend on where

V is on the circle



Pascal's Theorem

A generalization of Pappus's theorem

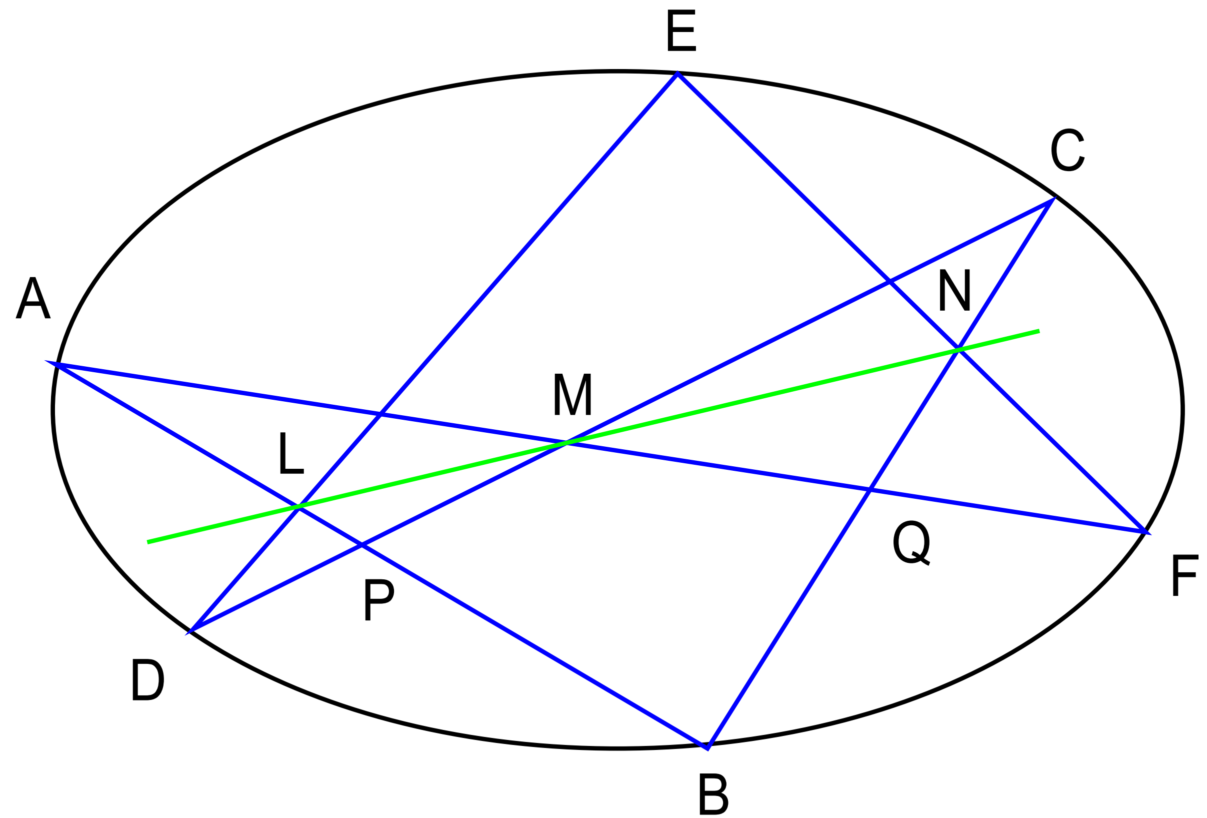
$$ALPB \overset{D}{\doublebarwedge} AECB$$

$$\overset{F}{\doublebarwedge} QNCB$$

B is common

$$ALP \overset{M}{\doublebarwedge} QNC$$

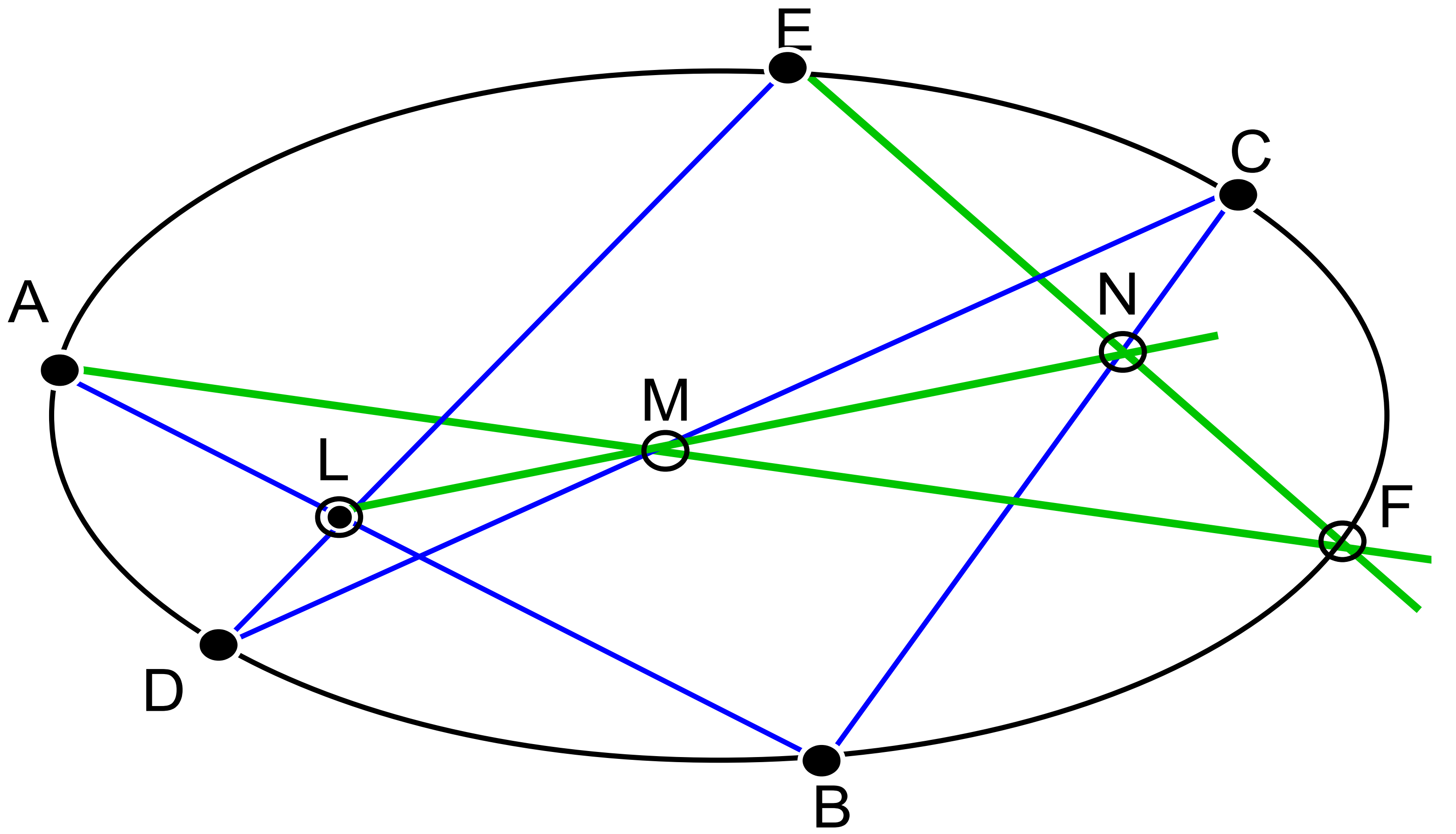
M lies on LN



A conic in projective geometry

One definition is by using Pascal's theorem

Five points define a conic

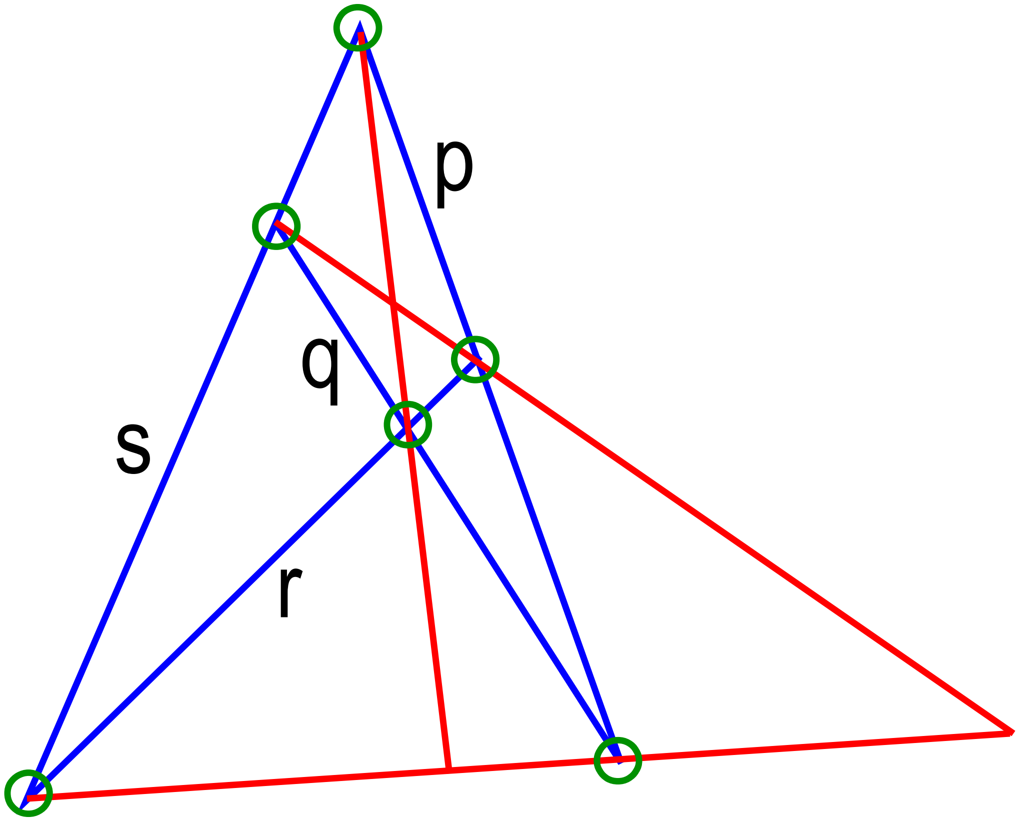


Axioms

- Any two lines have a single point in common
- Any two points have a single line in common
- There are four points with no three in a line
- A projectivity that leaves three points of a line invariant leave all of them invariant
- The diagonals of a complete quadrilateral are not on the same line

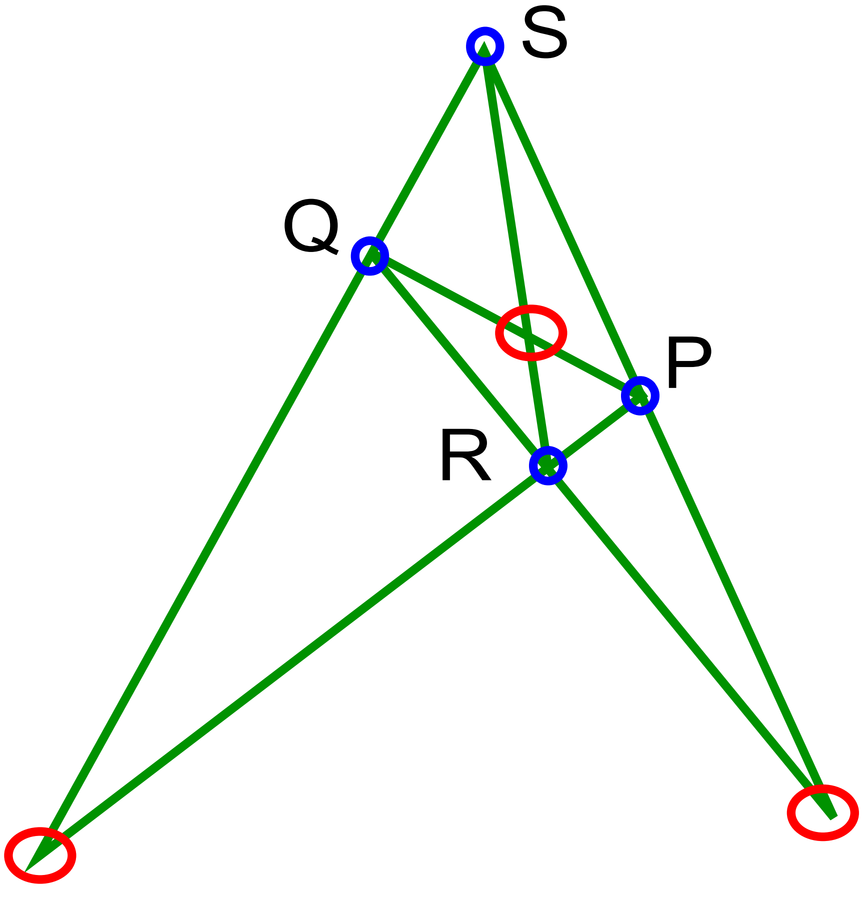
The complete quadrilateral

Defined by 4 lines



The complete quadrangle

Defined by 4 points



Barycentric coordinates

Very similar to homogenous or projective coordinates

(x, y, z) is the same as (kx, ky, kz)

Not all zero

Projective coordiantes use (x, y, 1) for (x, y)

and (x, y, 0) is a point at infinity

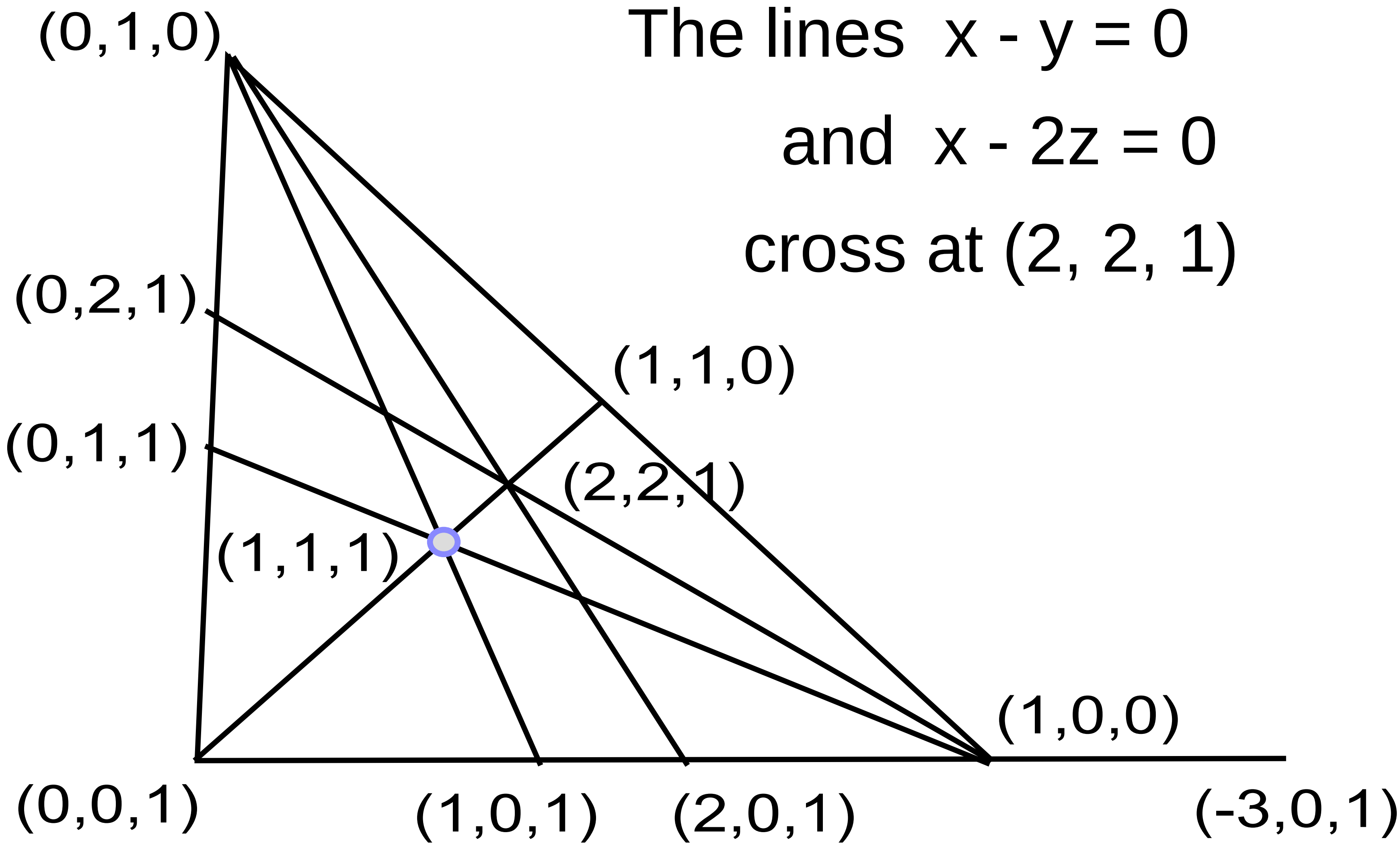
$ax+by+c = 0$ becomes $ax+by+cz = 0$

Barycentric points and lines

The lines x - y = 0

and x - 2z = 0

cross at (2, 2, 1)



Duality

Equation $ax + by + cz = 0$

is the same as $xa + yb + zc = 0$

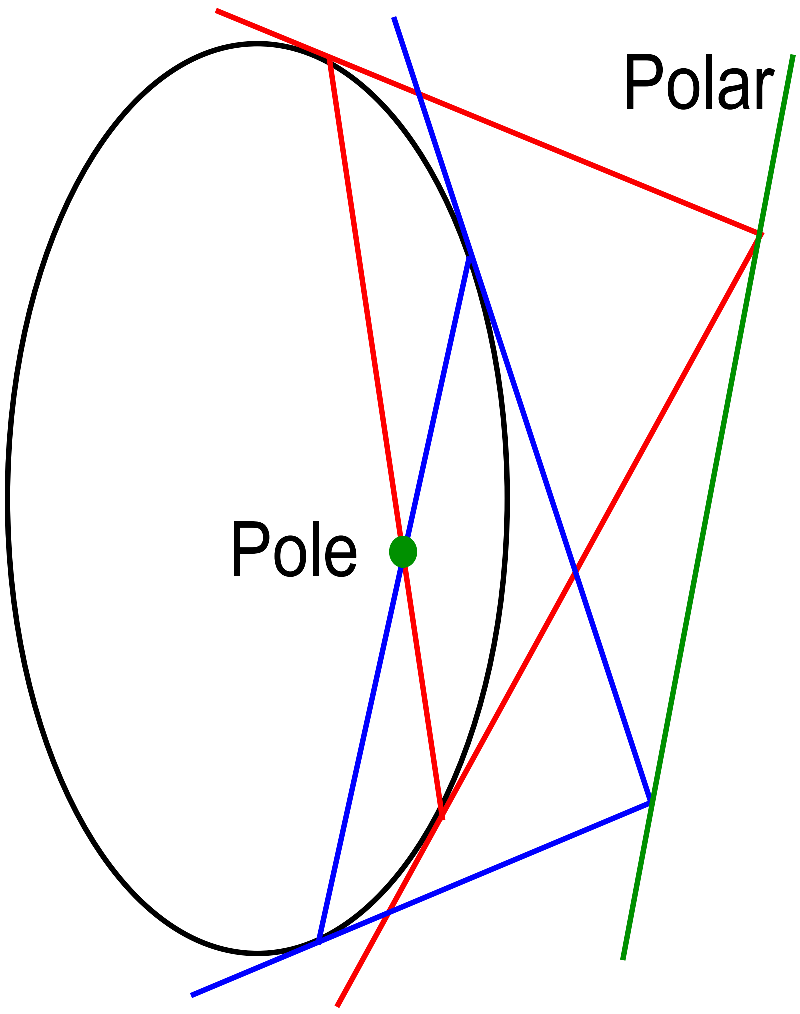
So the equation $1.x - 1.y + 0.z = 0$

Describes the line (1, -1, 0)

Points (1, 1, 0) and (1, 1, 1) give line (1, -1, 0)

Lines (1, -1, 0) and (1, 0, -2) give point (2, 2, 1)

Pole and Polar



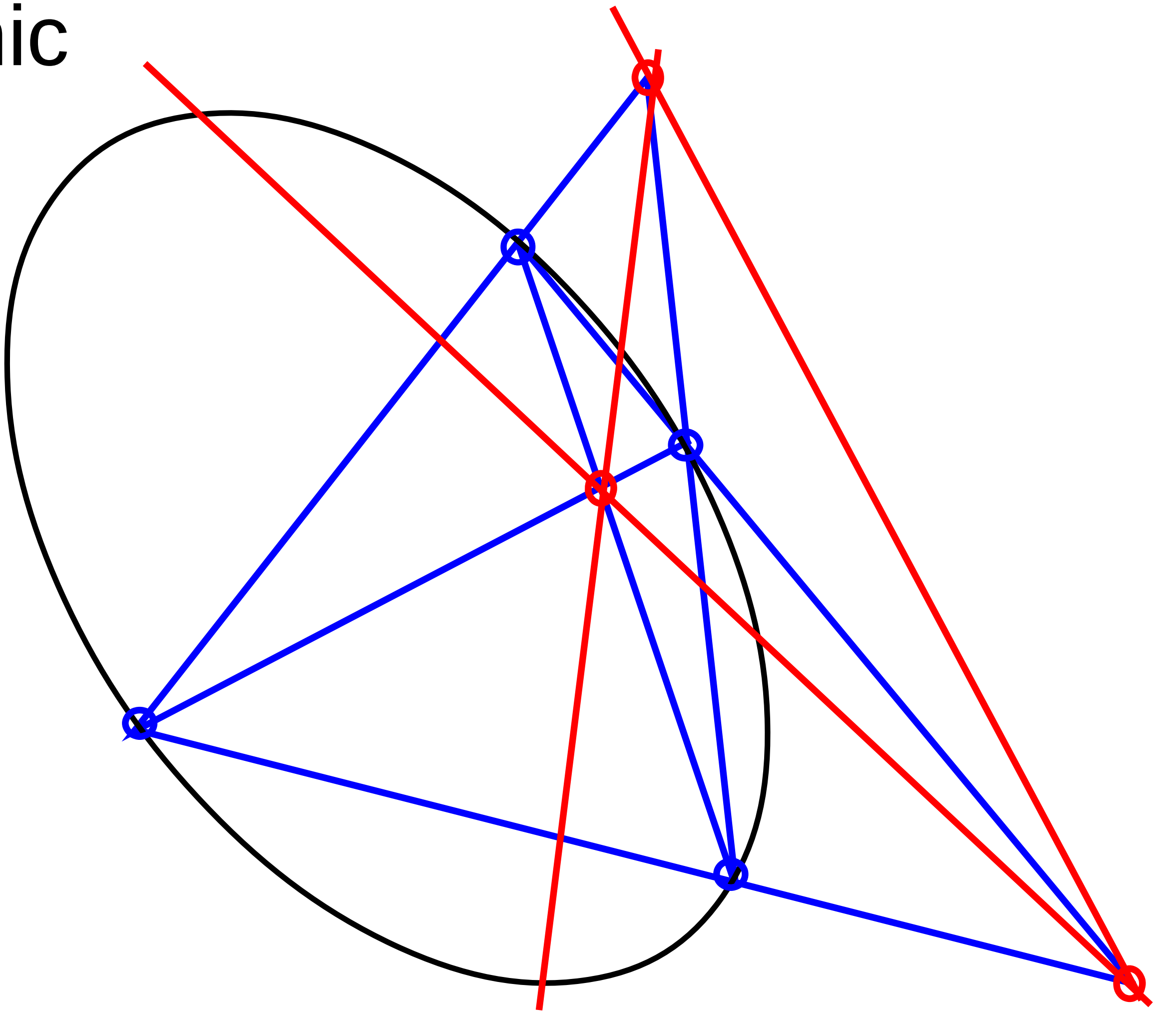
A correlation changes lines into points and points into lines

The mapping between poles and polars in Euclidean geometry does this and can be generalised to conics.

Another way to get the Polar

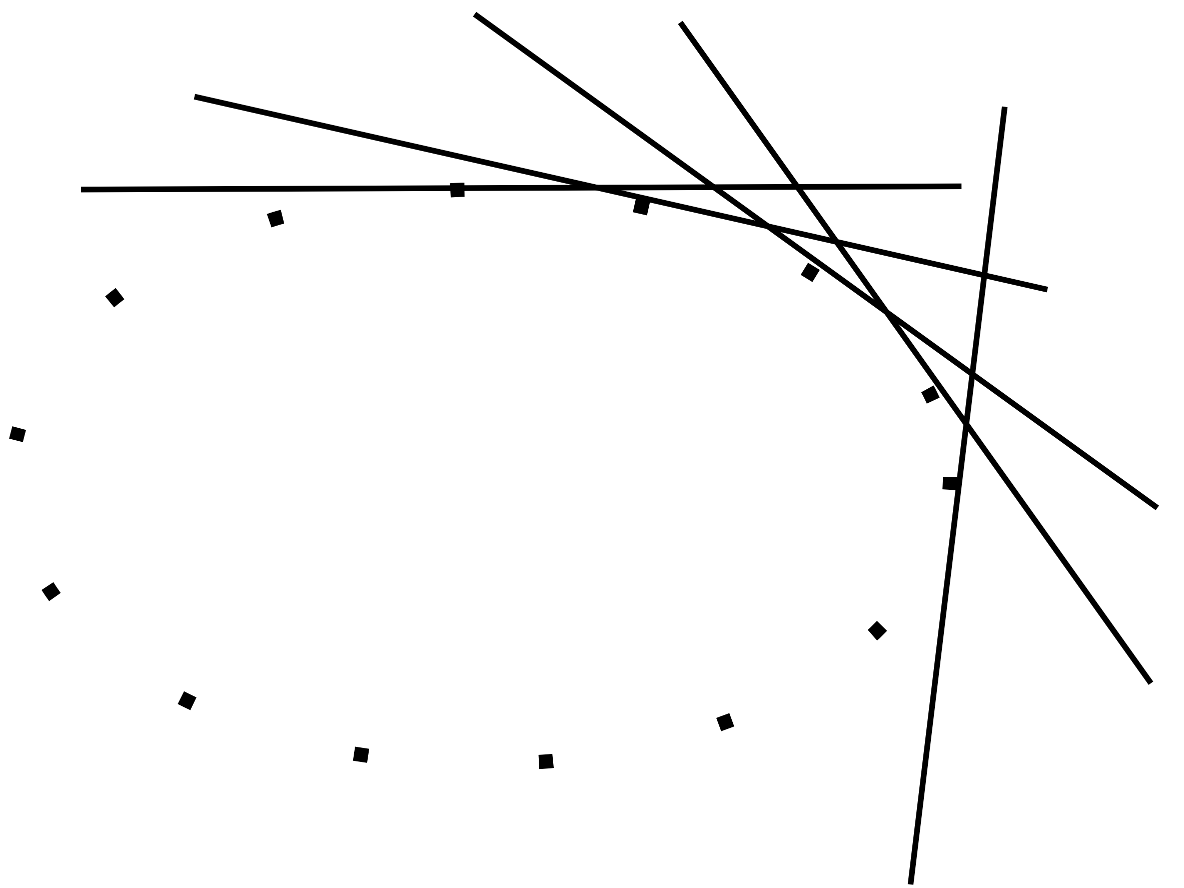
Draw two lines through the pole giving four points on the conic

- giving two more points which are on the polar.



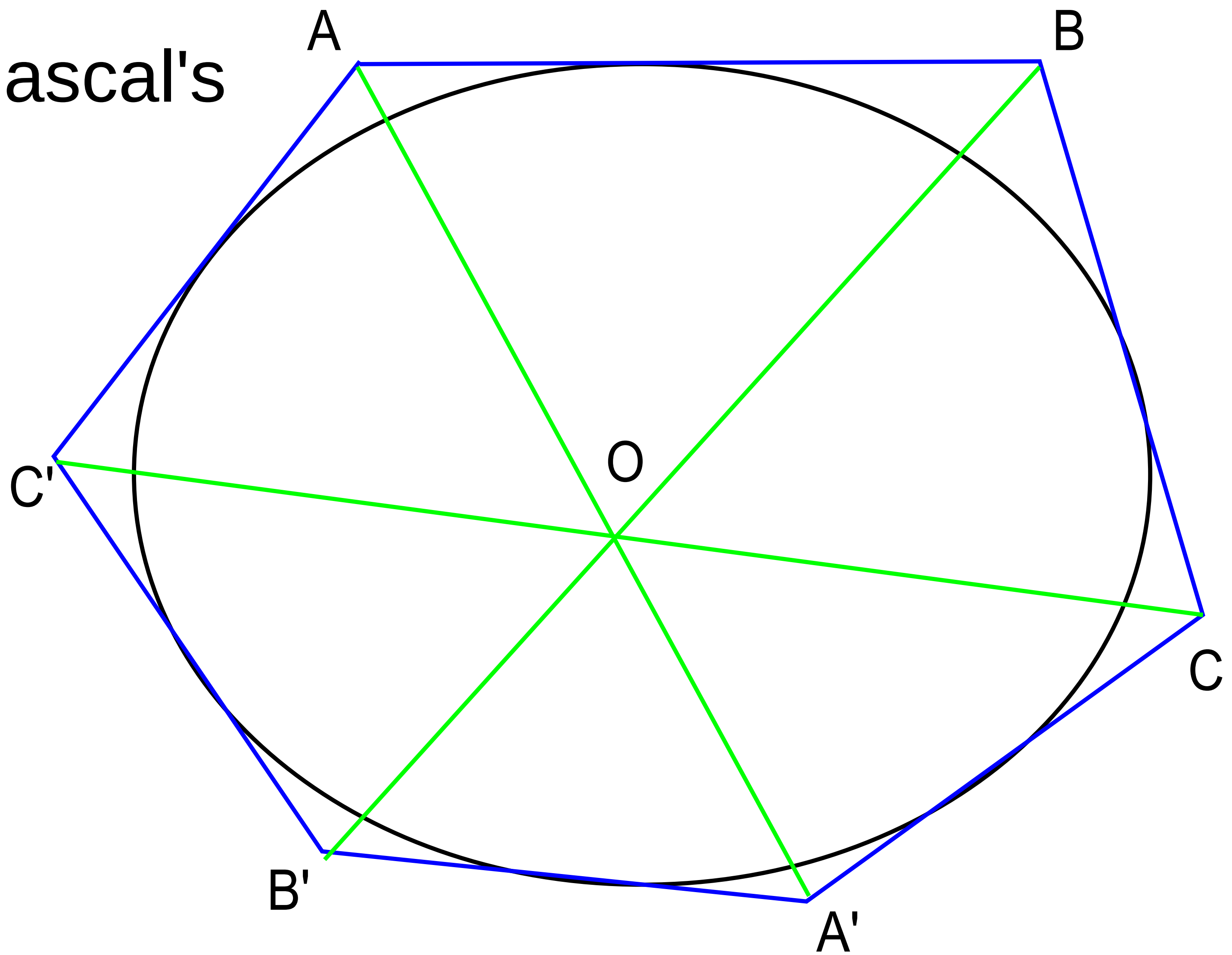
Another definition of a conic

Those points or lines which are incident with themselves under an order two correlation (one that is an identity when applied twice).



Briranchon's Theorem

Dual of Pascal's Theorem



Discrete Geometry

31 points and 31 lines ($5^2 + 5 + 1$)

6 points on every line

6 lines through every point

Can be given barycentric coordinates

(i mod 5, j mod 5, k mod 5)

Also by line l and point p are incident if

l+p =0, 1, 3, 8, 12, or 18 mod 31

Example in a finite geometry

Line l and point p are incident if

l+p =0, 1, 3, 8, 12, or 18 mod 31

The points 0, 4, 6, 9, 16, 17 are on a conic