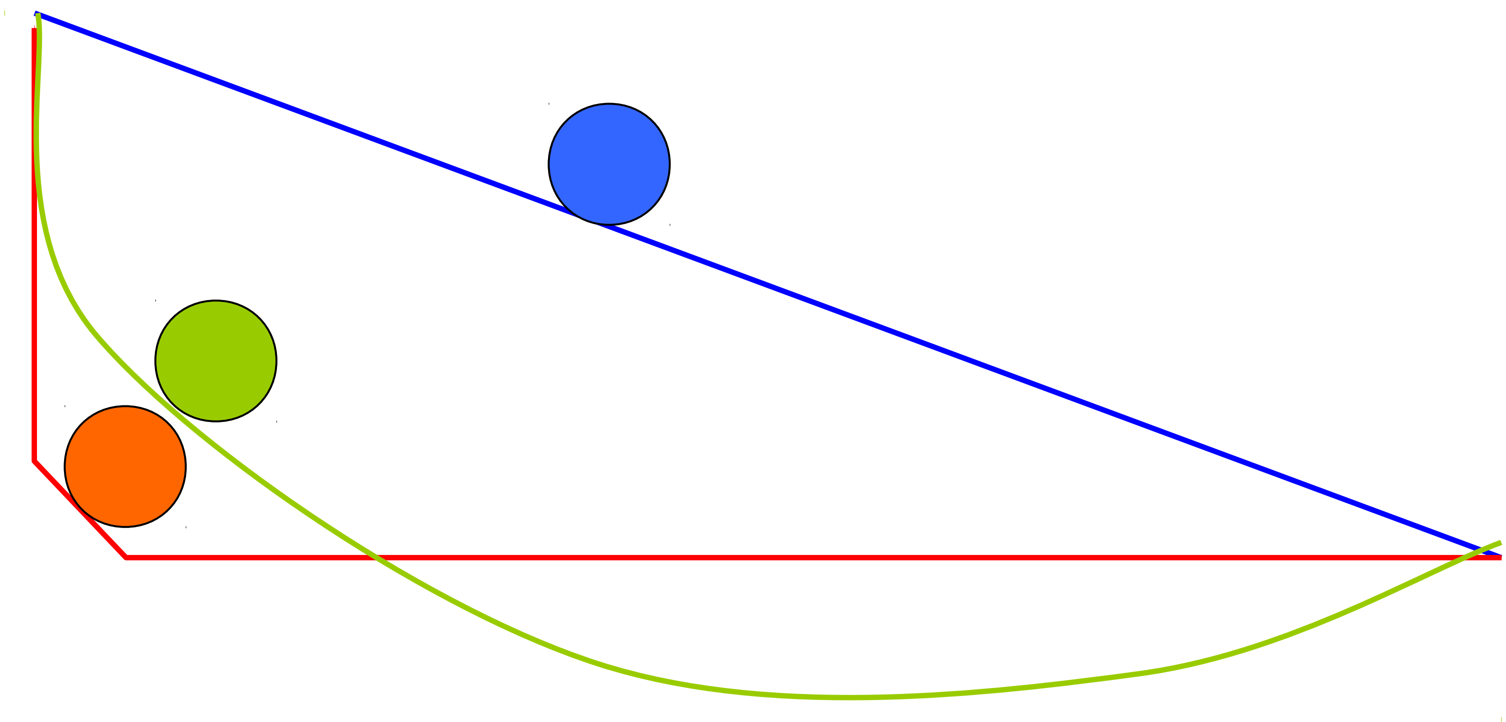
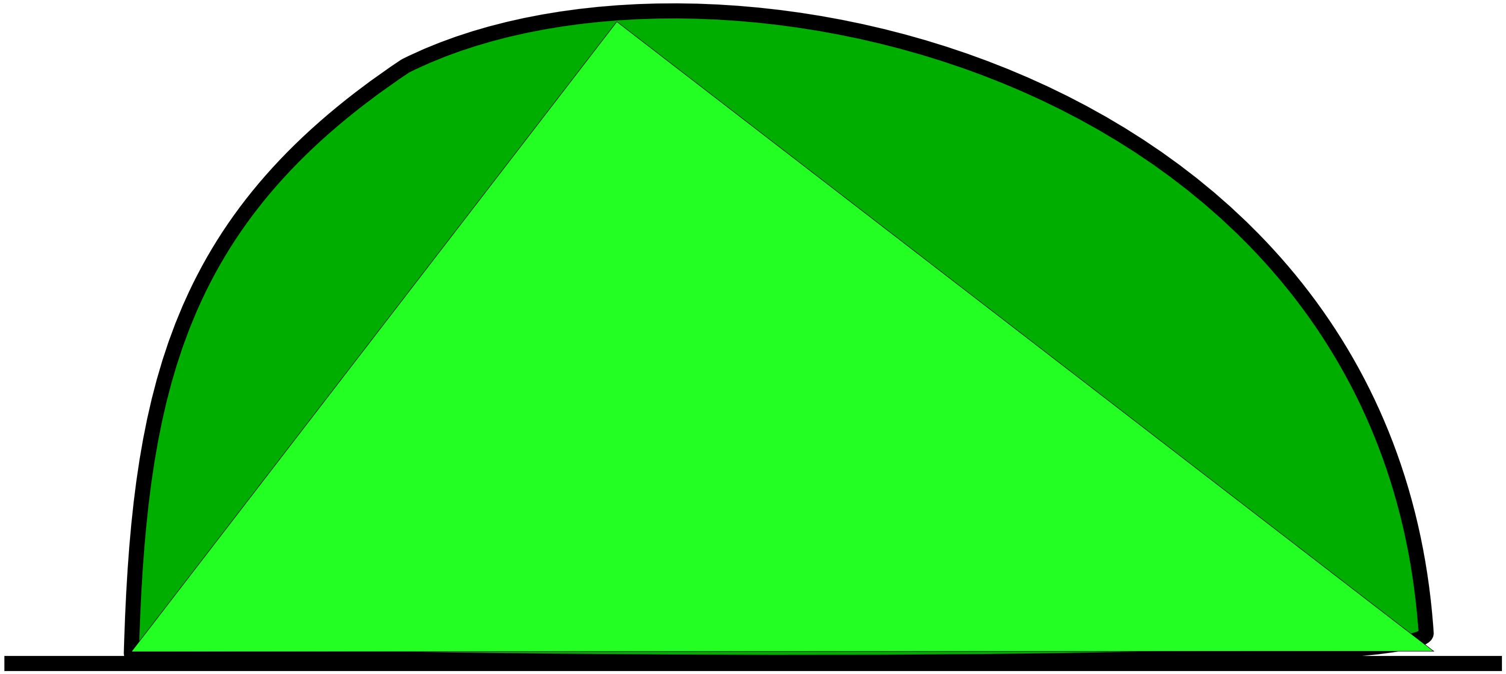
The fastest way down



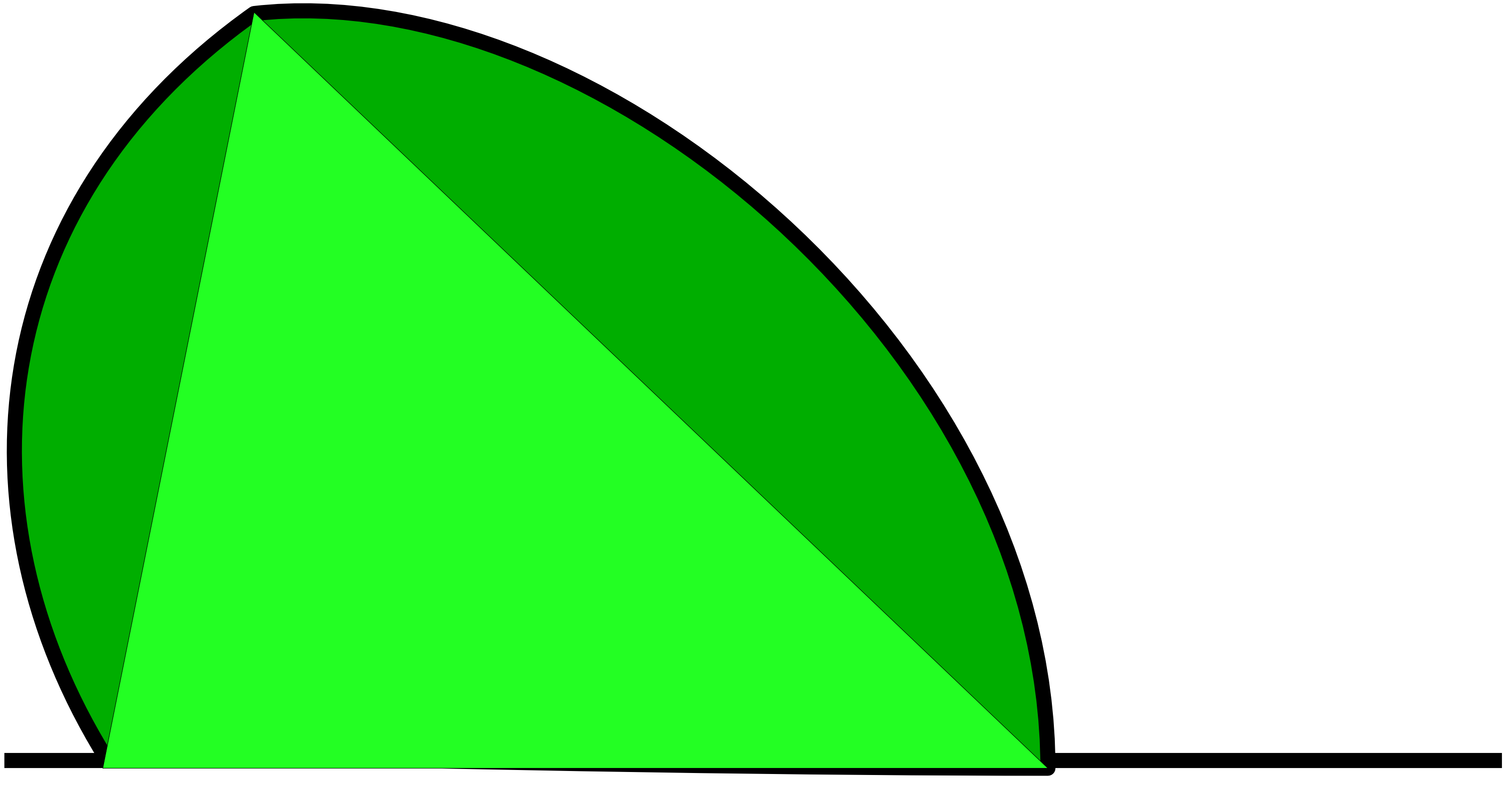
Queen Dido’s problem

Zenodorus 200BC, Pappus 300AD, Steiner 1841

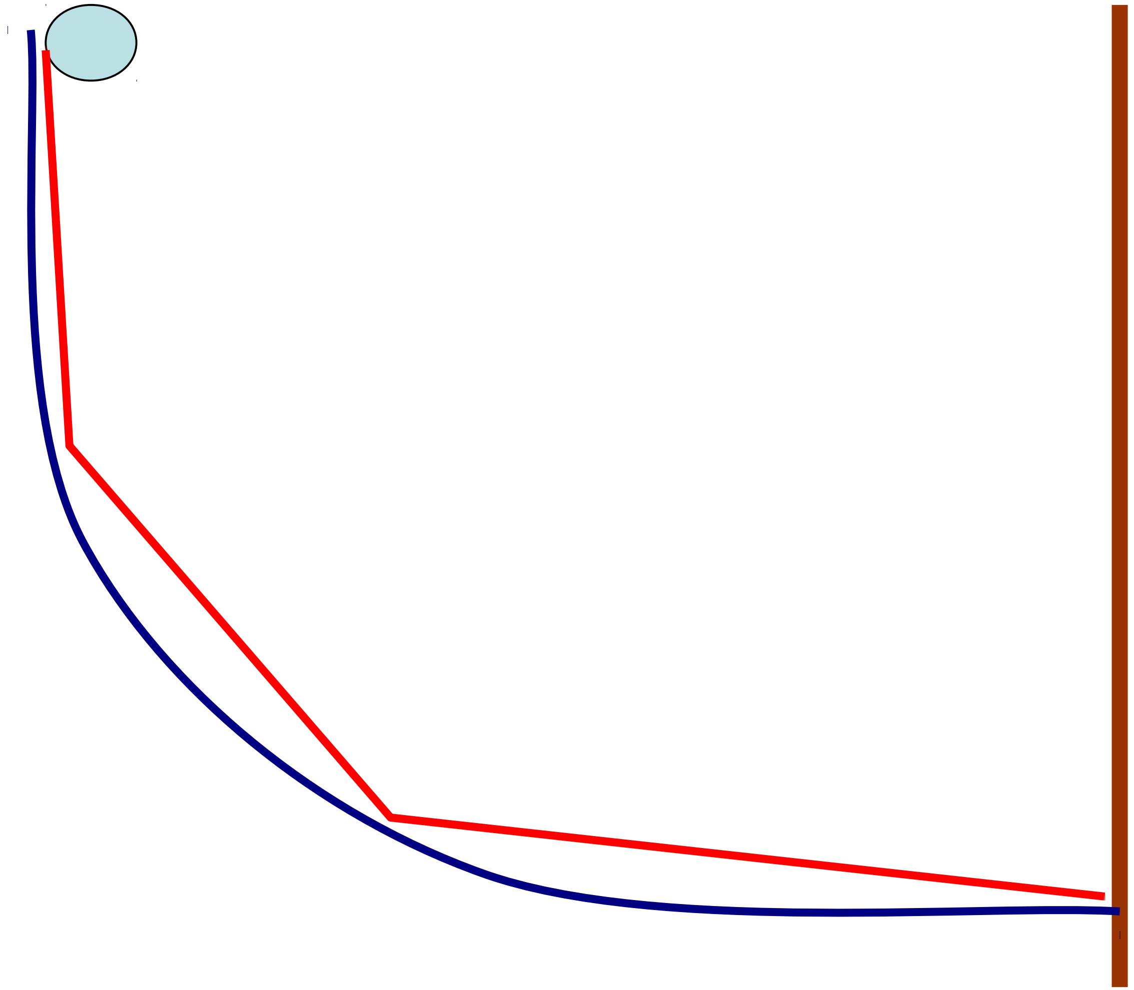
Queen Dido’s Problem



Queen Dido's Problem

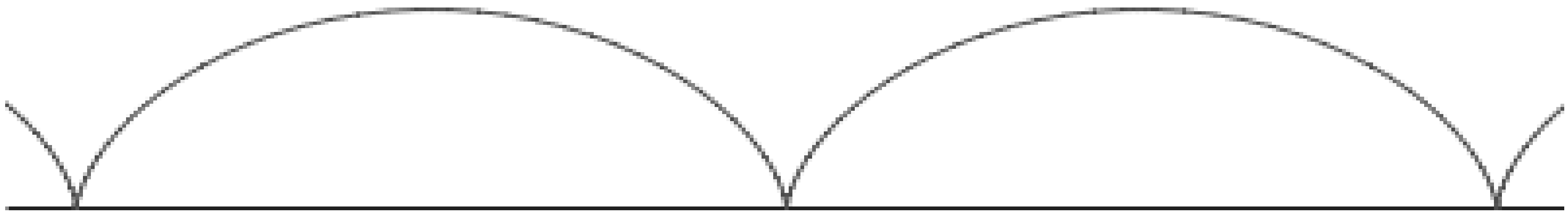


Galileo’s Scholium Problem



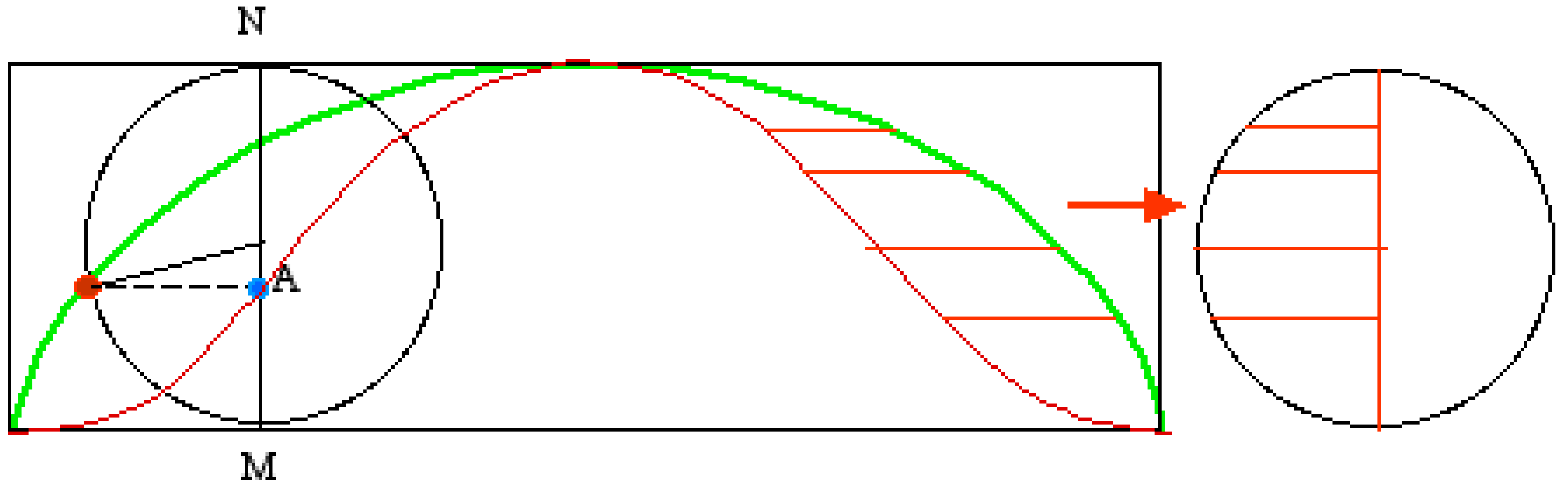
Cycloid

$$
\begin{aligned}
x &= r(\theta - \sin\theta) \\
y &= r(1 - \cos\theta)
\end{aligned}
$$



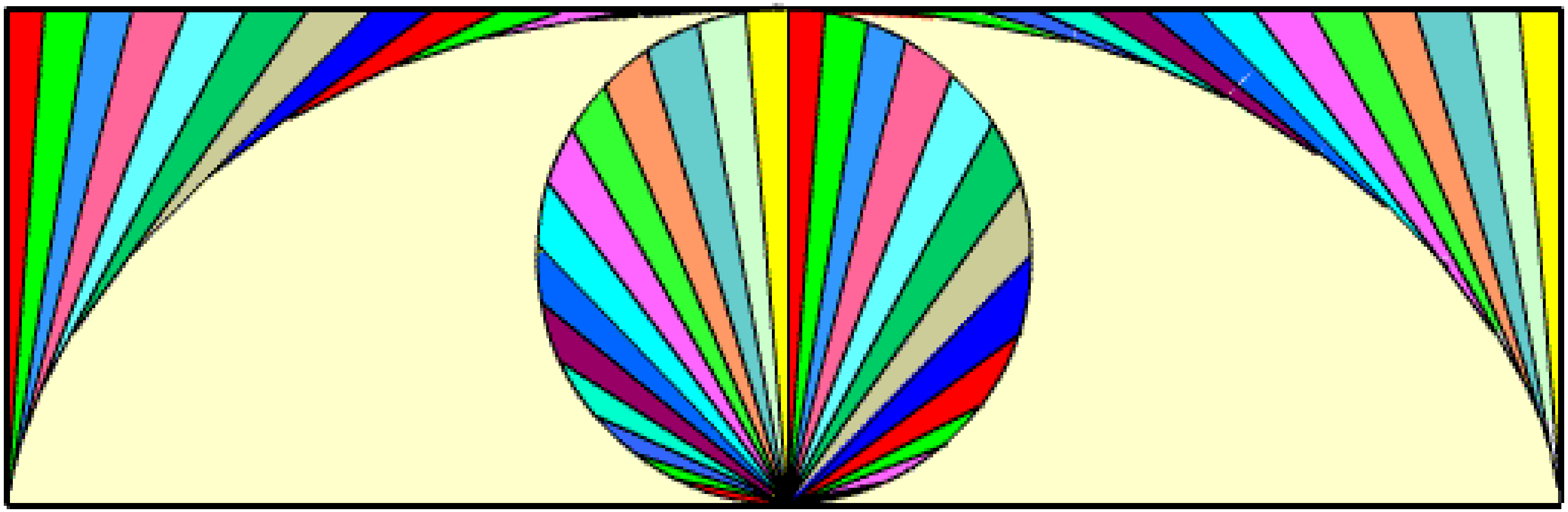
- First studied by Nicholas of Cusa and Charles Bouvelles
- Galileo called it a cycloid and tried to find the area of an arc. Area found by G.P. de Roberval, also Toricelli
- Length of arc found by Christopher Wren

Area of Cycloid



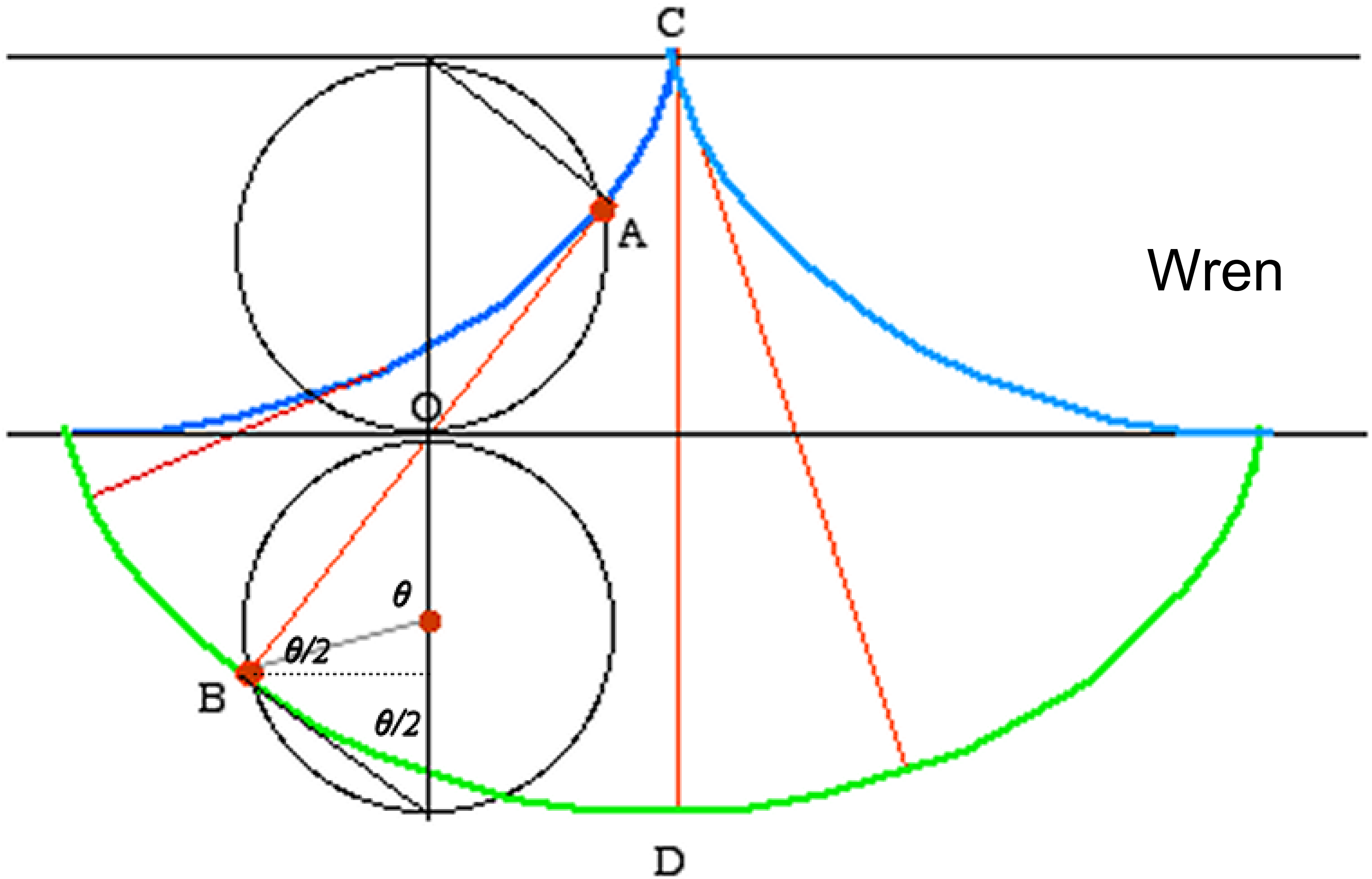
- Roberval and Toricelli
- Cavalieri's Principle

Visual Calculus by Mamikon



- The area of a tangent sweep is equal to the area of its tangent cluster, regardless of the shape of the original curve.

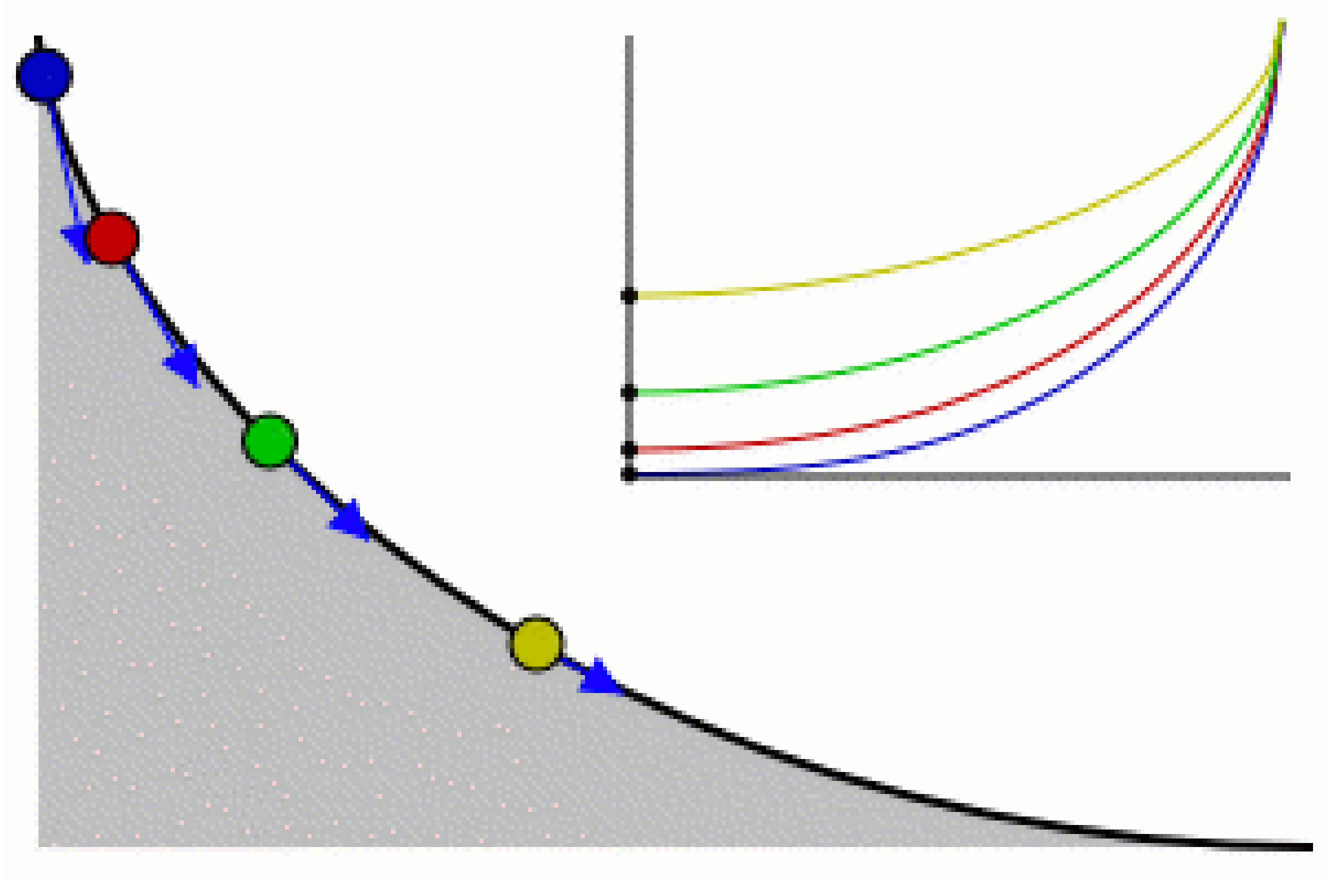
Length of cycloid



Wren

Tautochrone

- Proposed by Pascal, solved by Huygens



This diagram and some others are from Wikipedia

Tautochrone

For simple harmonic motion

$$s'' = -k^2 s \quad s = A \sin kt \quad T = 2\pi / k$$

Virtual gravity constraint

$$g \cos \frac{\theta}{2} = k^2 s \qquad \frac{g}{2} \sin \frac{\theta}{2} d\theta = -k^2 ds$$

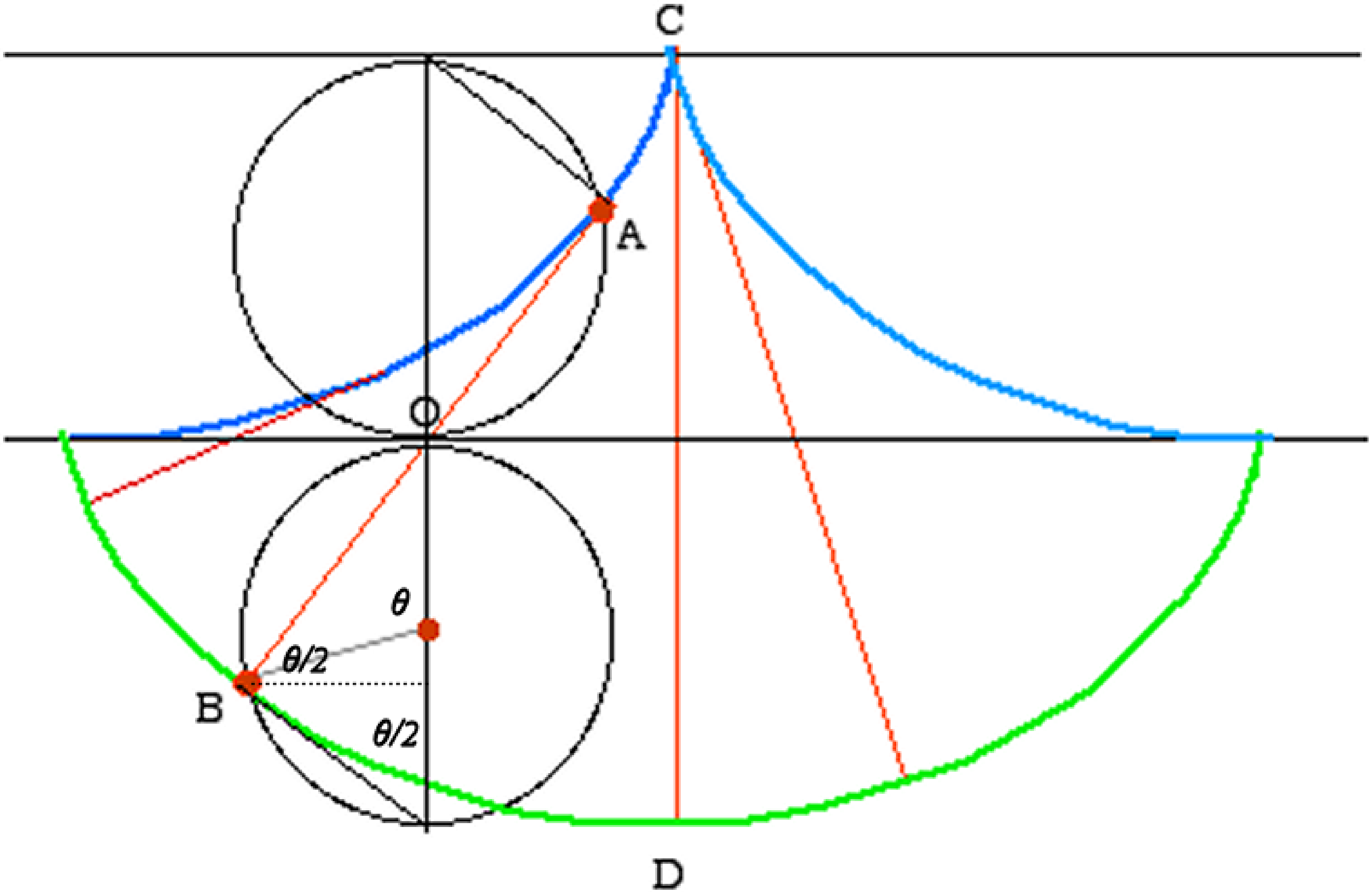
What happens with the cycloid

$$ds = -4rsin\frac{\theta}{2}d\frac{\theta}{2} \longrightarrow k = \sqrt{\frac{g}{4r}}$$

Brachistochrone

Brachistochrone

- Proposed + solved by Johann Bernoulli in 1696 using Fermat's principal of least time for light.
- Also Leibniz, l'Hôpital, Jakob Bernoulli, and Newton


C
A
O
θ
θ/2
B
θ/2
D

Brachistochrone

Fastest path can be split into sub-parts – dynamic optimisation

Can split into parts in a number of ways:

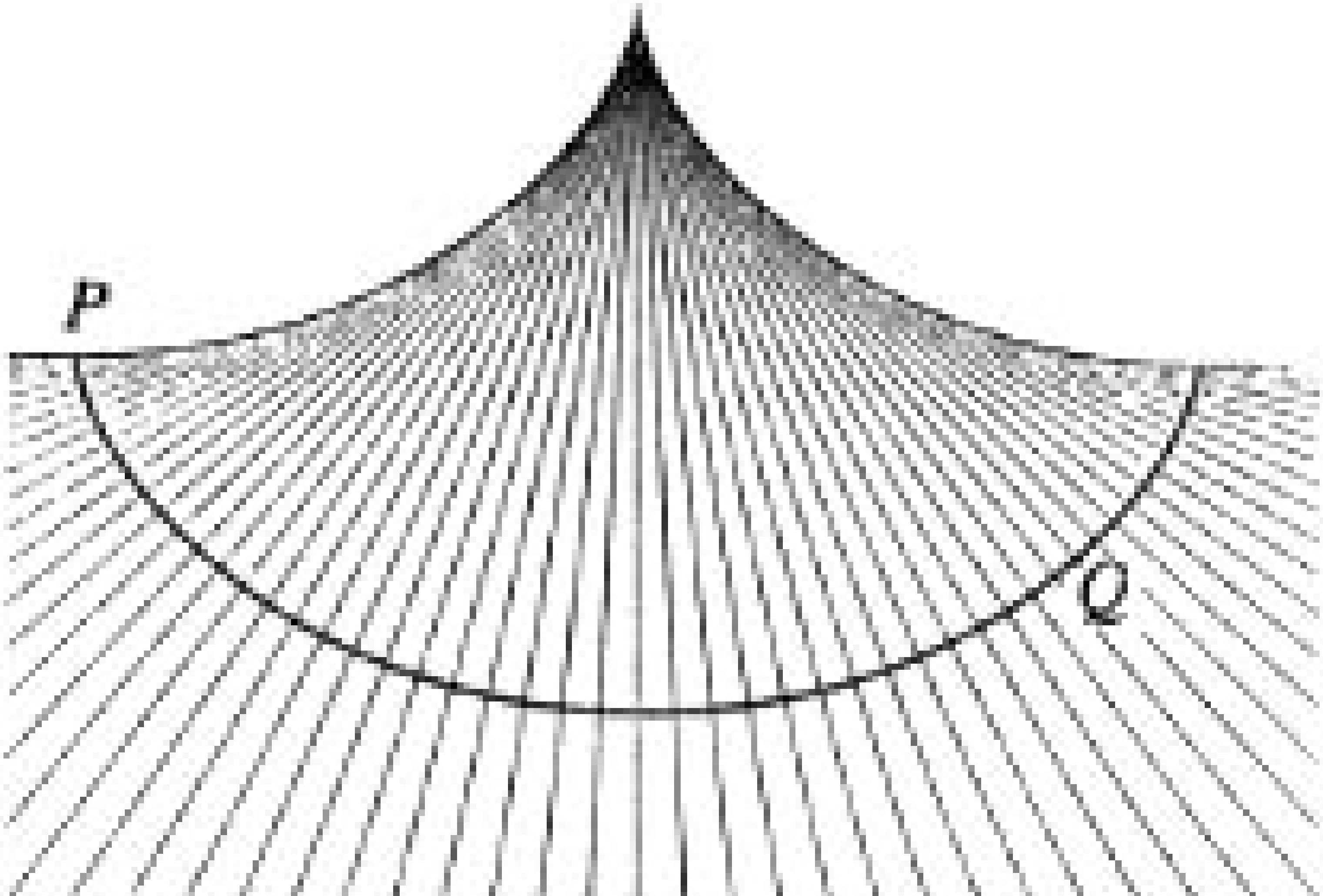
Solution using Wren's involute again.

$$v = \sqrt{2gh} \qquad h = 2r\cos^2(\theta/2)$$

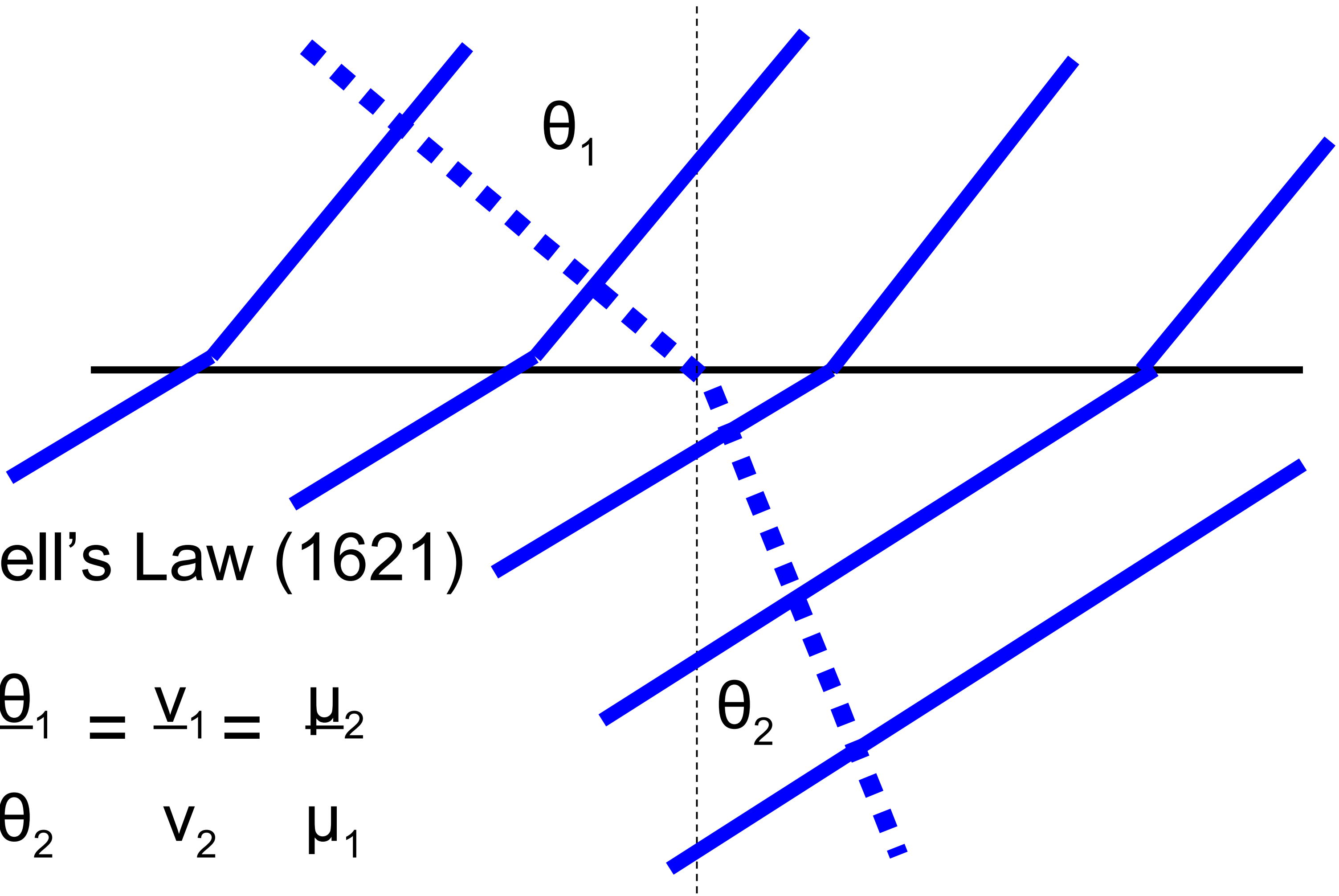
$$\frac{(d+x)\cos(\theta/2)}{\sqrt{2gx\cos^2(\theta/2)}} \cdot \frac{d\theta}{2} \qquad \frac{d+x}{\sqrt{x}} \qquad \frac{r}{\sqrt{x}} + \sqrt{x}$$

$$x = d$$

Brachistochrone



Path of light

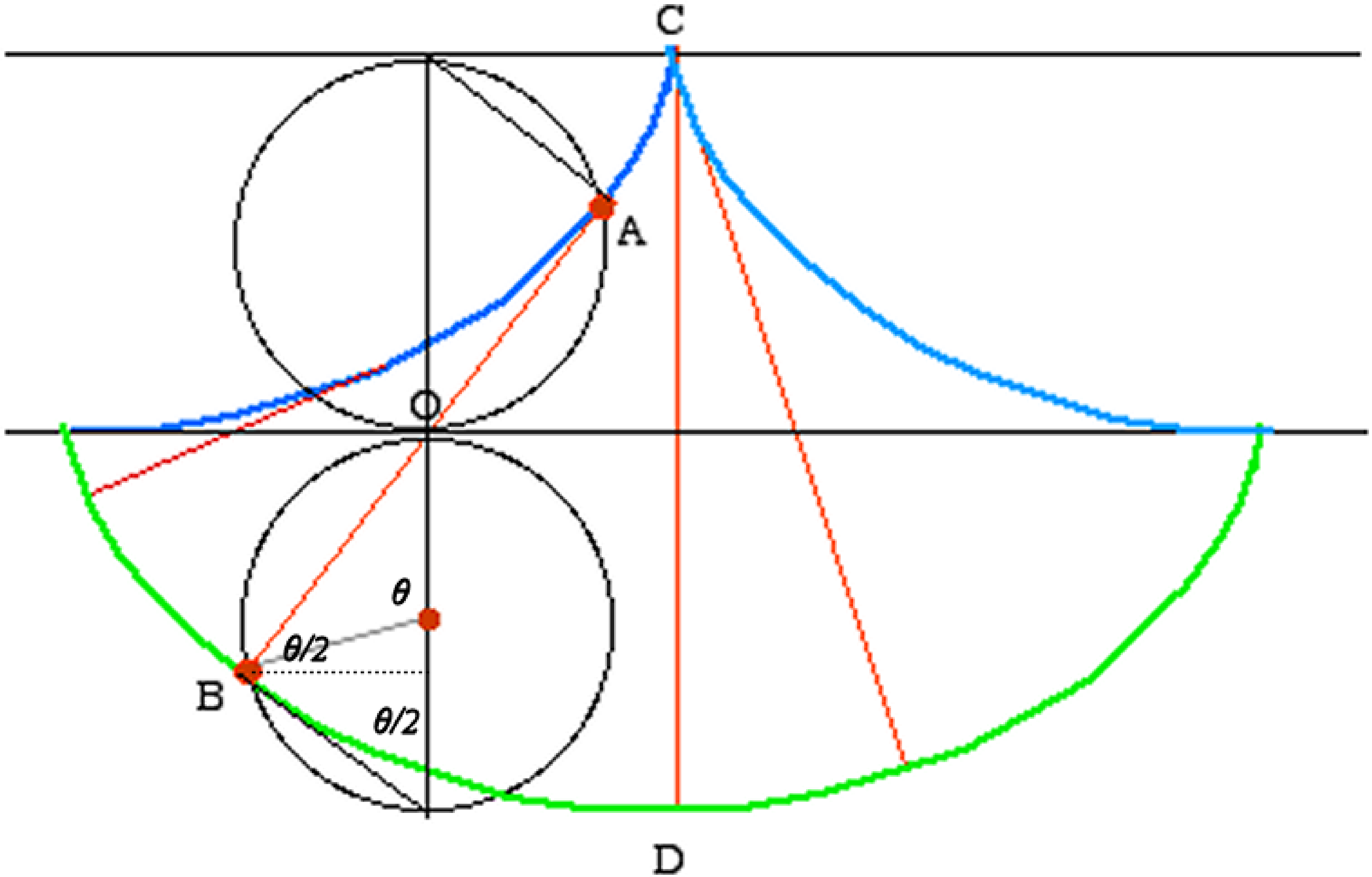


Snell's Law (1621)

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}$$

Fermats Principle

- The principle of least time in optics – the light from one point to another follows the path of least time. (Fermat 1662)
- Hero of Alexandria noticed it first for reflections,it was then extended for refraction by Alhacen in 1021.
- But Newton assumed light went faster in glass rather than slower.


C
A
O
θ
θ/2
B
θ/2
D

Solution using Fermat's principle and horizontal slices

$$v = \sqrt{2gh}$$

$$\frac{\sin\phi}{v} = K \qquad \frac{\sin\phi}{\sqrt{-2gy}} = \frac{1}{\sqrt{2gD}}$$

For cycloid

$$\frac{\cos(\theta/2)}{\sqrt{2g \cdot 2r\cos^2(\theta/2)}} = \frac{1}{\sqrt{4gr}}$$

Principle of last action

- Solutions of physical problems with no dissipation involve stationary solutions of the integral of the Lagrangian which is the Kinetic energy minus the potential energy.
- A global optimisation problem

Euler-Lagrange Equation

- A central equation in the Calculus of Variations. Devised in the 1750s.
- Turns the least action global optimisation problem into a differential equation.
- Under fairly general conditions a stationary condition is satisfied at each point of a solution, except if the solution goes along a boundary .

Euler-Lagrange Equation

$$J = \int_a^b L(x, f(x), f'(x))\, dx$$

$$f(a) = c, f(b) = d$$

$$g_\epsilon(x) = f(x) + \epsilon\eta(x) \qquad \eta(a) = \eta(b) = 0$$

$$J(\epsilon) = \int_a^b L(x, g_\epsilon(x), g'_\varepsilon(x))\, dx$$

$$\frac{\mathrm{d}J}{\mathrm{d}\varepsilon} = \int_a^b \frac{\mathrm{d}L}{\mathrm{d}\epsilon}(x, g_\varepsilon(x), g'_\varepsilon(x))\, dx$$

$$\frac{\mathrm{d}J}{\mathrm{d}\epsilon} = \int_a^b \left[\eta(x)\frac{\partial L}{\partial g_\varepsilon} + \eta'(x)\frac{\partial L}{\partial g'_\varepsilon} \right] dx$$

Stationary

$$J'(0) = \int_a^b \left[\eta(x)\frac{\partial L}{\partial f} + \eta'(x)\frac{\partial L}{\partial f'} \right] dx = 0$$

$$0 = \int_a^b \left[\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} \right] \eta(x)\, dx + \left[\eta(x)\frac{\partial L}{\partial f'} \right]_a^b$$

$$0 = \int_a^b \left[\frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'} \right] \eta(x) \, dx$$

The Euler-Lagrange equation

$$0 = \frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'}$$

$$\frac{\delta J}{\delta y} = L_y - \frac{d}{dt} L_{y'}$$

Brachistochrone

Vertical slices + E-L equation

$$\int_{p1}^{p2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

$$\longrightarrow \quad \frac{1}{\sqrt{1+y'^2}\sqrt{2gy}} = C$$

Via Beltrami identity
when no dependence on x

$$L - f'\frac{\partial L}{\partial f'} = C$$

Lavrentiev phenomenon

- Hilbert first to give good conditions for E-L to give a stationary solution.
- Convex area, *L* positive C^3, homogenous in the deriviatives $\rightarrow$ countable sections of paths that are solutions of E-L or go along boundary
- Stationary path not solution for instance

$$L(t, x, x') = (x^3 - t)^2 x'^6$$

$$x(0) = 0, \;\; x(1) = 1$$

Where is it now?

- An ancient problem which has developed over the ages. Queen Dido's problem. Brachistochrone, Tautochrone.
- The basis of modern mechanics. Euler-Lagrange Equation. Hamiltonians. Statistical mechanics
- Still under active development. Noethers theorem. Quantum mechanics. Measurement theory?

Noether’s Theorem

- Any symmetry of the action implies a conservation law

Extreme Physical Information

- John Archibald Wheeler: *All things physical are information-theoretic in origin and this is a participatory universe... Observer participancy gives rise to information; and information gives rise to physics.*

Cramér-Rao inequalty

- Fisher Information measures the information a random variable carries about an unknown parameter.
- The variance of any unbiased estimator is at least as high as the inverse of the Fisher information

$$\mathrm{E}\left[\hat{\theta}(X)-\theta\right]=\int\left[\hat{\theta}(X)-\theta\right]\cdot f(X;\theta)\,dx=0$$

$$\frac{\partial}{\partial\theta}\mathrm{E}\left[\hat{\theta}(X)-\theta\right]=\int\left(\hat{\theta}-\theta\right)\frac{\partial f}{\partial\theta}\,dx-\int f\,dx=0$$

$$\int\left(\left(\hat{\theta}-\theta\right)\sqrt{f}\right)\left(\sqrt{f}\,\frac{\partial\ln f}{\partial\theta}\right)\,dx=1$$

$$\left[\int\left(\hat{\theta}-\theta\right)^2 f\,dx\right]\cdot\left[\int\left(\frac{\partial\ln f}{\partial\theta}\right)^2 f\,dx\right]\geq 1$$