Knot Theory

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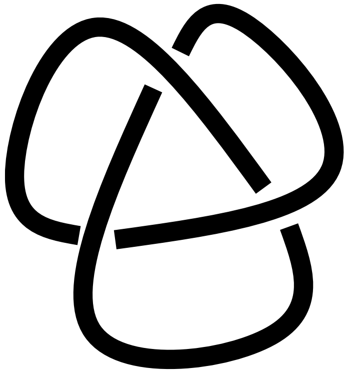
Topics

- Definitions
- Reidemeister moves
- Knot Invariants
- Extensions

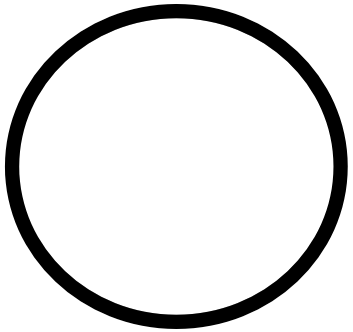
Definitions

- A knot: A closed non-self intersecting curve in three dimensions. Problem with infinite numbers of turns, called wild knots. We'll just consider tame knots.



Trefoil

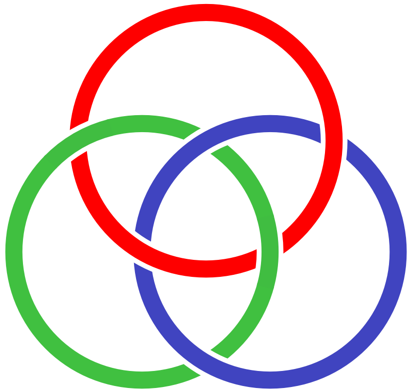
- The unknot – a not knotted knot



Unknot

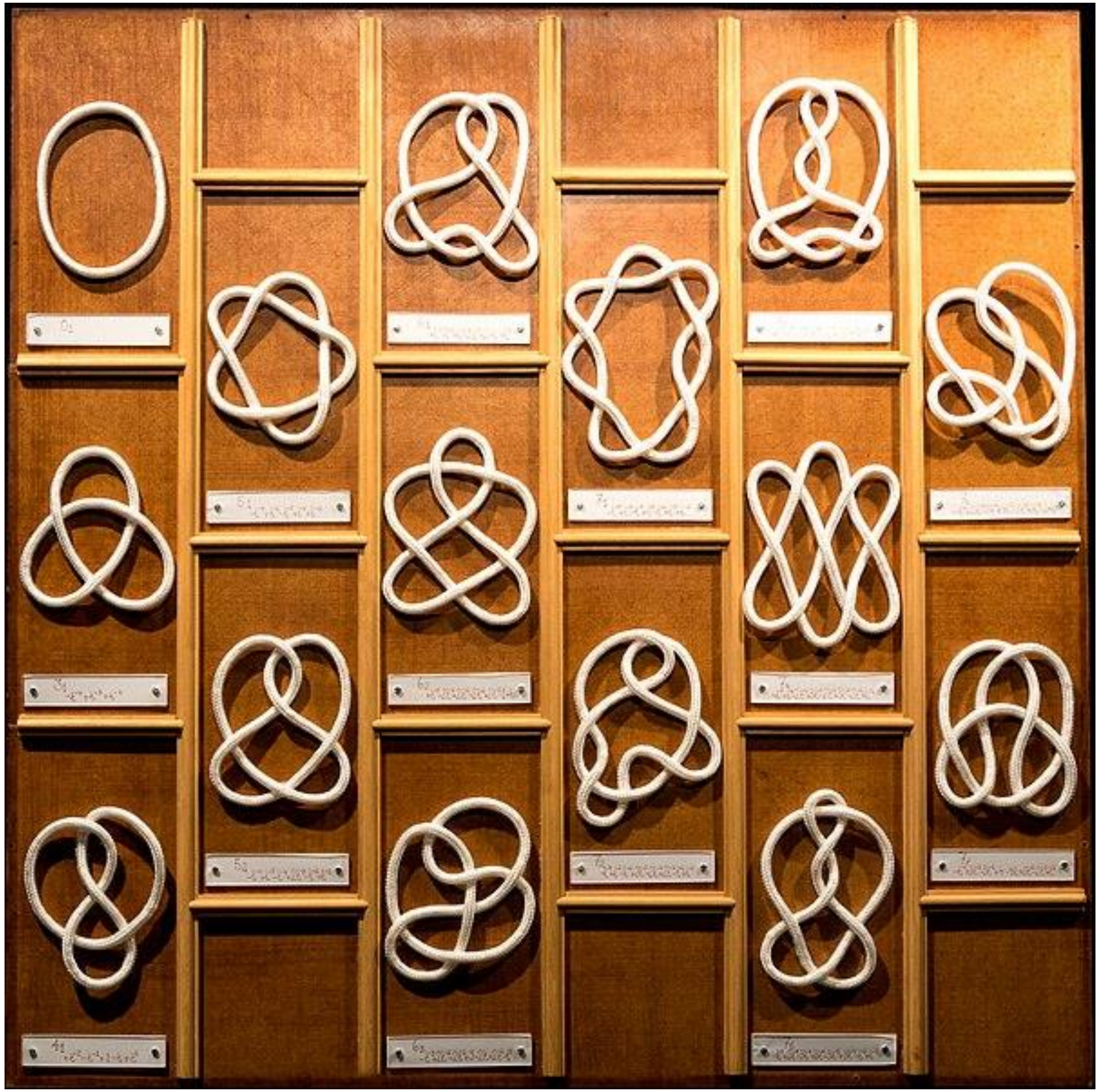
- A link: A collection of one or more knots

  We won't be considering them much

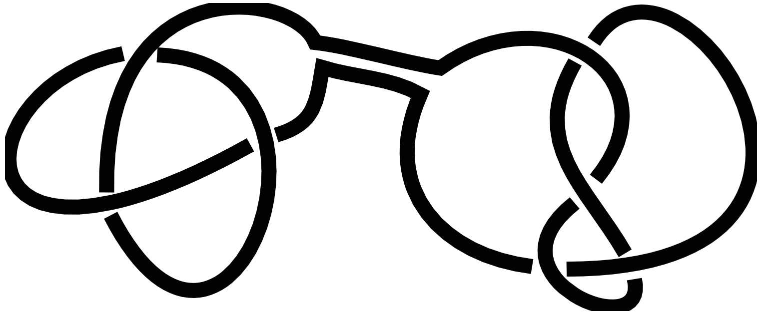


Borromean Rings

- A projection – these pictures in 2D

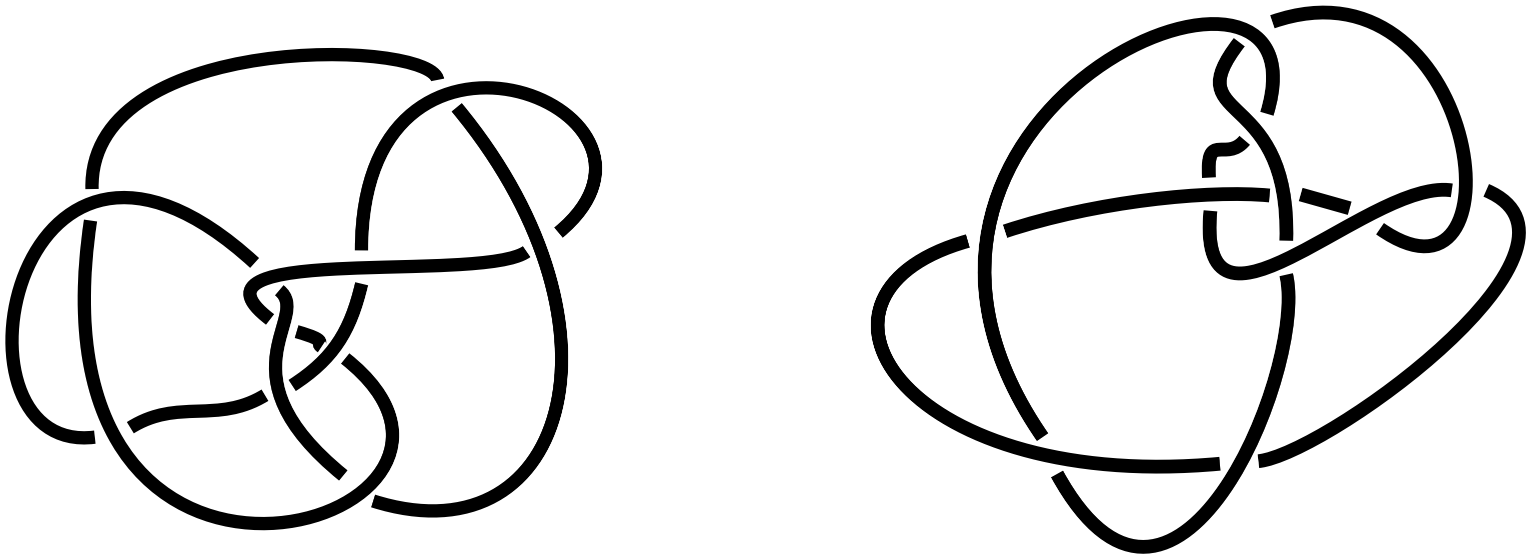


Prime knots (not a knot sum) with up to 7 crossings, excluding any mirror pairs



A knot sum. The reef knot and granny knot are knot sums

The Perko Pair

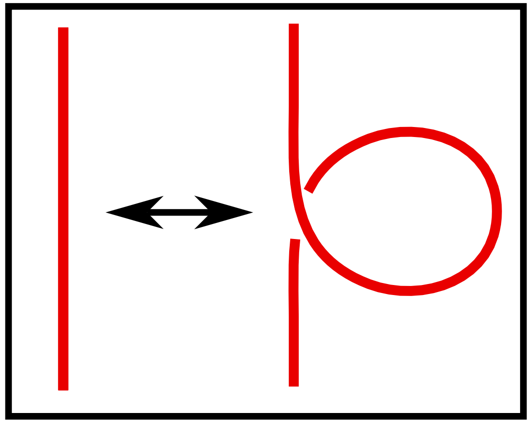


Listed in old tables of knots up to ten crossings. Only found to be the same in 1973 by Kenneth Perko.

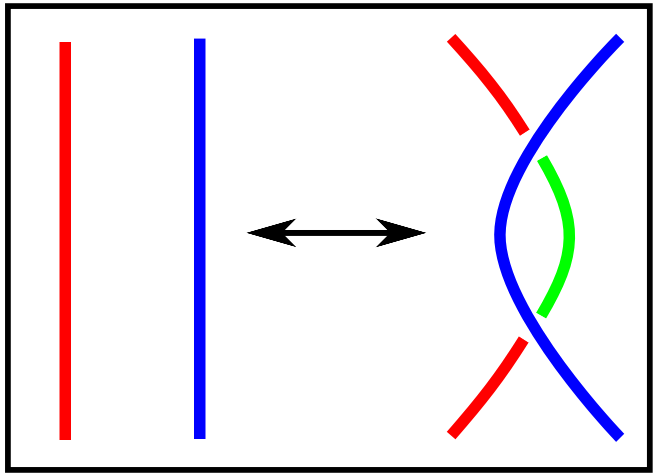
The big problem of Knot Theory – find a simple way to classify knots so they can be told apart, or even to be the unknot!

Reidemeister moves

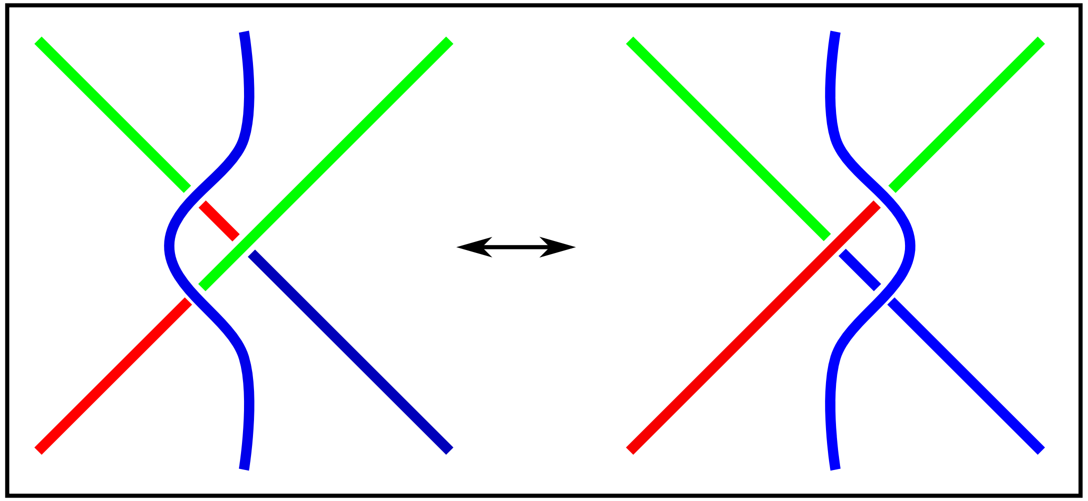
Operations on a projection onto the plane of a knot.



Type 1



Type 2



Type 3

Every operation on a knot can be expressed as a combination of these
(but not every knot can be three-coloured!)

Knot Invariants

If some property computed for a knot is invariant if any of the Reidemeister moves is done then it is a knot invariant.

If two knots have different values of a knot invariant they are different.

However if it is the same they are not necessarily the same.

The minimum number of crossings is a knot invariant
but is not easy to compute.

The most useful ones are straightforward to compute from any projection though it may be quite tedious.

Tricolourability

Is the trefoil the unknot?

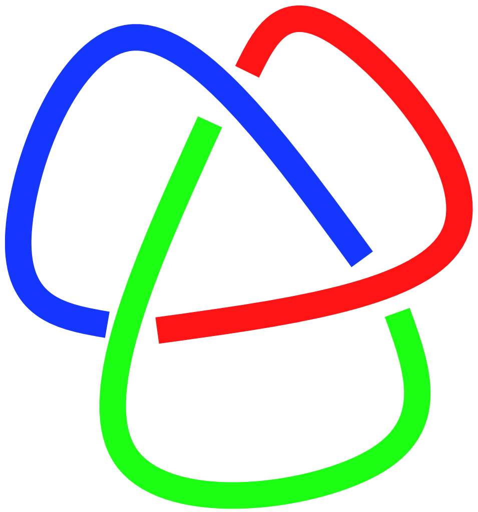
A knot is tricolourable if it can be coloured using three colours such that at every crossing either just one colour or all three are used.

The illustration of the Reidemeister moves showed how this property is preserved.

And a trefoil is tricolourable

Therefore the trefoil cannot be turned into the unknot. QED

But we can't yet tell it from its mirror image.

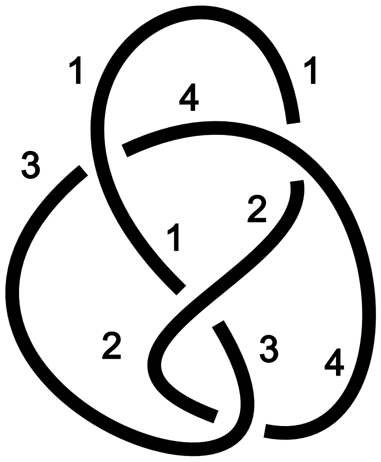


p-colourability generalises tricolourability. At each crossing there are three segments and we have an equation

$$2x-y-z = 0 \pmod p$$

Where x is the 'colour' of the upper segment and y and z are the 'colours' of each side of the lower segment, and x,y,z are in 0..p-1.

A knot can be n-coloured if this can be done and more than one colour is used. A figure-8 knot isn't tricolourable but is 5-colourable.



With a=1, b=2, c=3, d=4

$$2a-c-d = -5$$
$$2d-a-b = 5$$
$$2b-a-c = 0$$
$$2c-b-d = 0$$

In fact if we put this as a matrix then p must divide the value of any determinant of a minor. (any line sums to zero so two can be removed as follows)

$$\begin{array}{c|ccc|}
2 & 0 & -1 & -1 \\
-1 & -1 & 0 & 2 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1
\end{array} \quad = \quad 5$$

p-colourability is fairly weak and doesn’t distinguish mirror images.

Not that that matters here as the figure-8 knot is the same as its mirror image! Try it!

Knot Polynomials

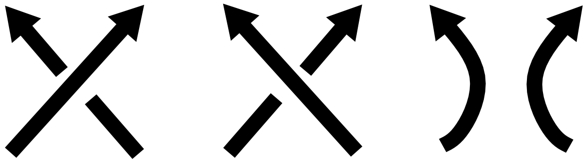
Knot polynomials are knot invariants which are polynomials.

The first one discovered was the Alexander polynomial devised by James Waddell Alexander II in 1923. John Conway devised a version of it called the Alexander Conway polynomial in 1969 using a skein relation and this led to the discovery of the Jones polynomial in 1984.

So firstly, what is a skein relation?

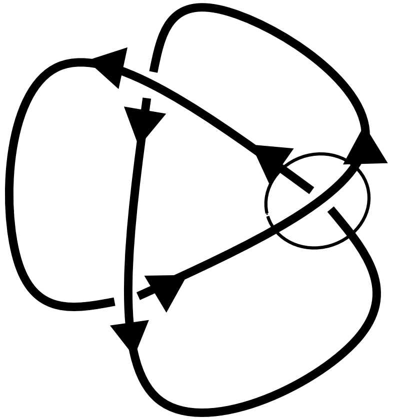
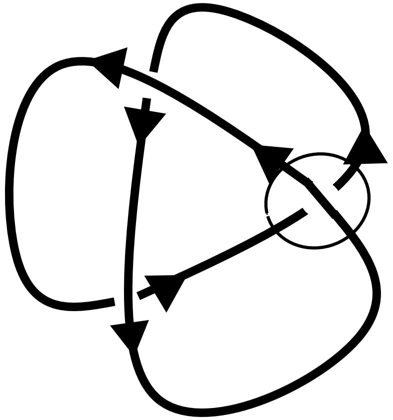
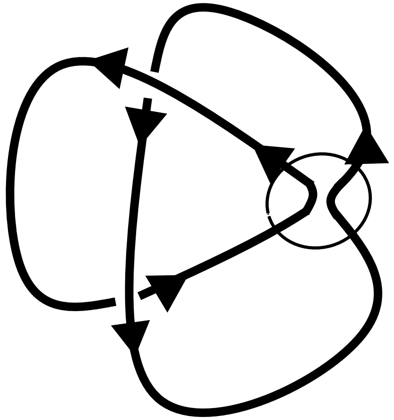
Skein relations

A recursive way of determining knot polynomials using a relation between three links which differ only at a crossing. For the Jones polynomial a knot is oriented and these are as follows



L_+ L_- L_0

As in

A.P() + B.P() + C.P() = 0

Subject to the rules

P(0) = 1   Applied to the unknot the polynomial is 1

And A, B, C are carefully chosen so Reidemeister moves doesn't change the result.

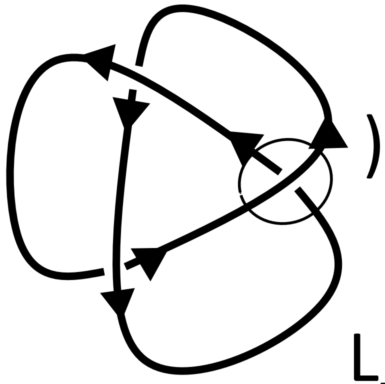
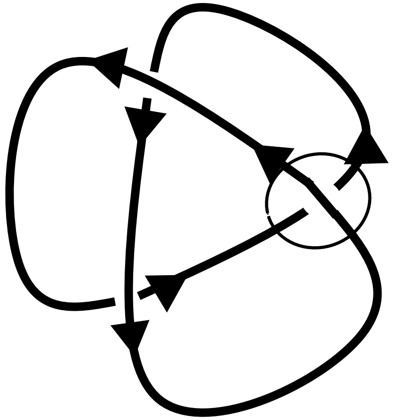
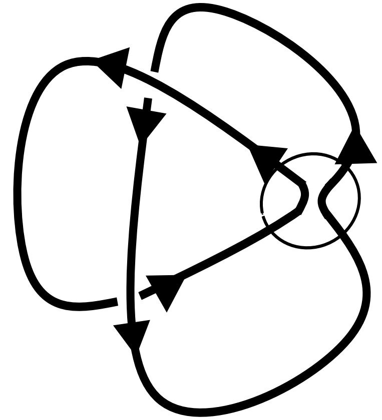
Well actually the first Reidemeister move does change it for the Bracket polynomial and the final result needs to be multiplied by a factor dependent on something called the writhe to get an invariant!

But we'll do the Jones Polynomial instead

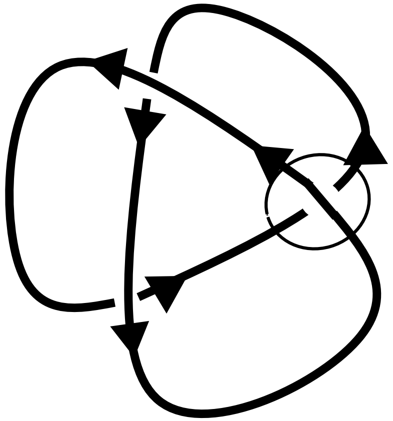
For the Jones polynomial denoted by V

$$(t^{1/2} - t^{-1/2})\, V(L_0) = t^{-1}\, V(L_+) - t\, V(L_-)$$

The right trefoil knot

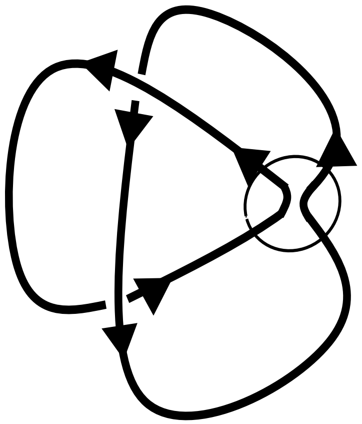
$t^{-1}\,V($  $) = t\,V($  $) + (t^{1/2} - t^{-1/2})\,V($  $)$

L_+ L_- L_0



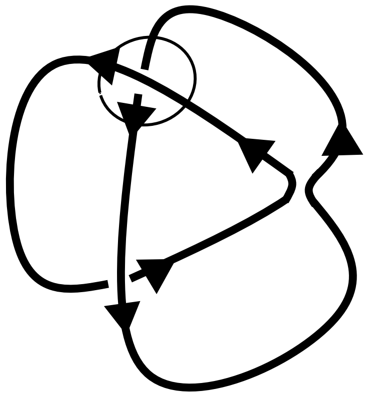
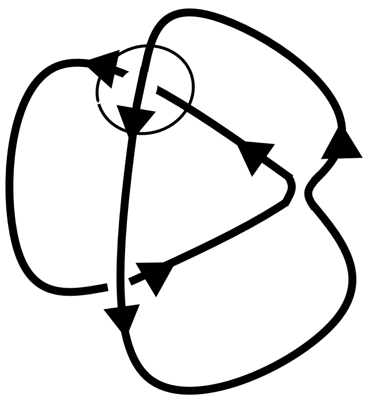
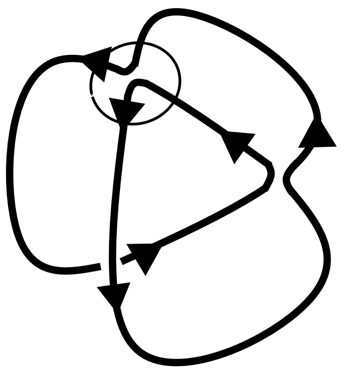
is the unknot

V(0) = 1



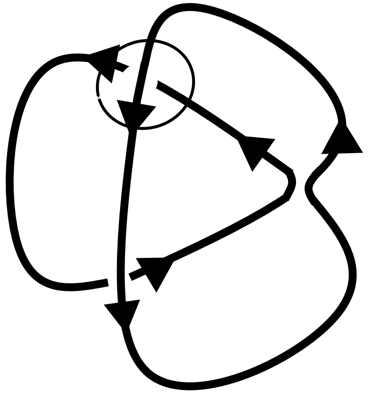
is two linked ones

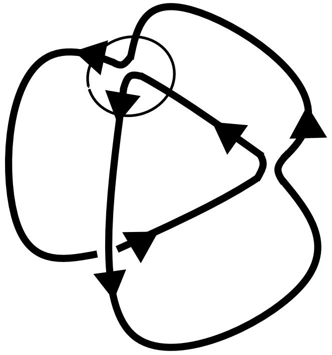
Two linked unknots - clockwise Hopf link

t^{-1} V() = t V() + $(t^{1/2} - t^{-1/2})$ V()

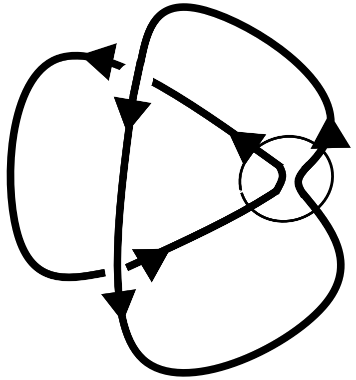
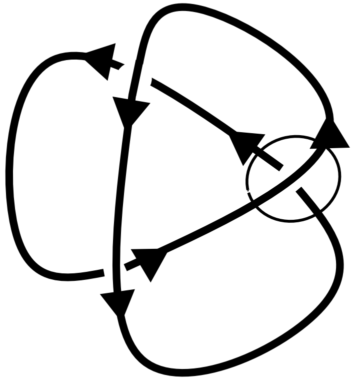
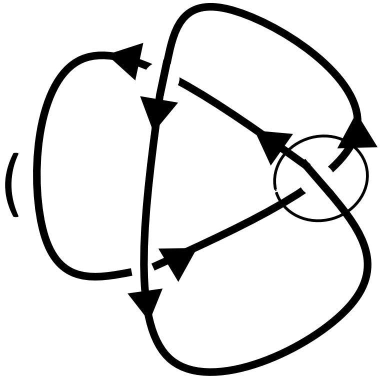
L_+ L_- L_0

The Jones polynomial depends on whether they go clockwise or anticlockwise round one another. This one is clockwise.

 is two unlinked unknots

 is the unknot

Two unlinked unknots

$(t^{1/2} - t^{-1/2})\ V($  $) = t^{-1}\ V($  $) - t\ V($  $)$

L_0 L_+ L_-

The right hand side knots are the unknot

$(t^{1/2} - t^{-1/2})\ V($ two unlinked unknots $) = t^{-1}\ V(0) - t\ V(0)$

Finally...

$V(0) = V(\text{unknot}) = 1$

$(t^{1/2} - t^{-1/2})\, V(\text{two unlinked unknots}) = t^{-1} V(0) \;-\; t\, V(0)$

$V(\text{two unlinked unknots}) = -\, t^{1/2} - t^{-1/2}$

$t^{-1} V(\text{two linked unknots}) = t\, V(\text{two unlinked unknots}) + (t^{1/2} - t^{-1/2})V(0)$

$V(\text{two linked unknots}) = t^2(-\, t^{1/2} - t^{-1/2}) + t\,(t^{1/2} - t^{-1/2}) = -\, t^{5/2} - t^{1/2}$

$t^{-1} V(\text{right trefoil}) = t\, V(0) + (t^{1/2} - t^{-1/2})\, V(\text{two linked unknots})$

$V(\text{right trefoil}) = t^2 - t^4 - t^2 + t^3 + t = -\, t^4 + t^3 + t$ Whew!

Mirror image trefoils are different

The mirror image of a knot has a Jones polynomial got by substituting 1/t for t, so

$V(\text{ left trefoil }) = -t^{-4} + t^{-3} + t^{-1}$

And since Reidemeister moves do not change the Jones polynomial the left handed trefoil cannot be turned into a right handed trefoil.

Extension from colourability

The Jones polynomial is stronger than p-colourability as substituting -1 for t gives plus or minus the determinant used to tell if a knot is p-colourable.

In the case of the trefoil $-t^4 + t^3 + t^1$ evaluated at -1 gives -3.

The Jones polynomial of the figure-8 knot is

$$t^2 - t + 1 - t^{-1} + t^{-2}$$

Which evaluates to 5 at -1 and it is 5-colourable

Extensions

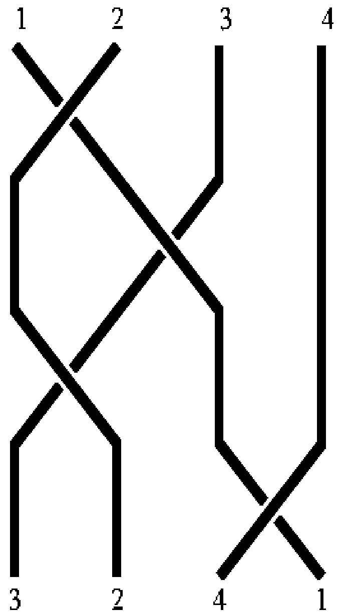
It is unknown whether the Jones polynomial always distinguishes a knot from the unknot.

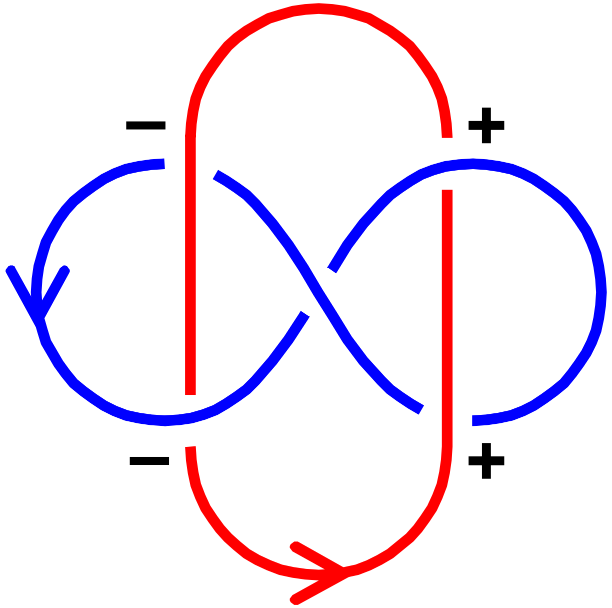
Lots of new knot invariants like that have been found. The Khovanov homology and knot Floer homology are known to always distinguish a knot from the unknot – but I haven’t worked out how they work!

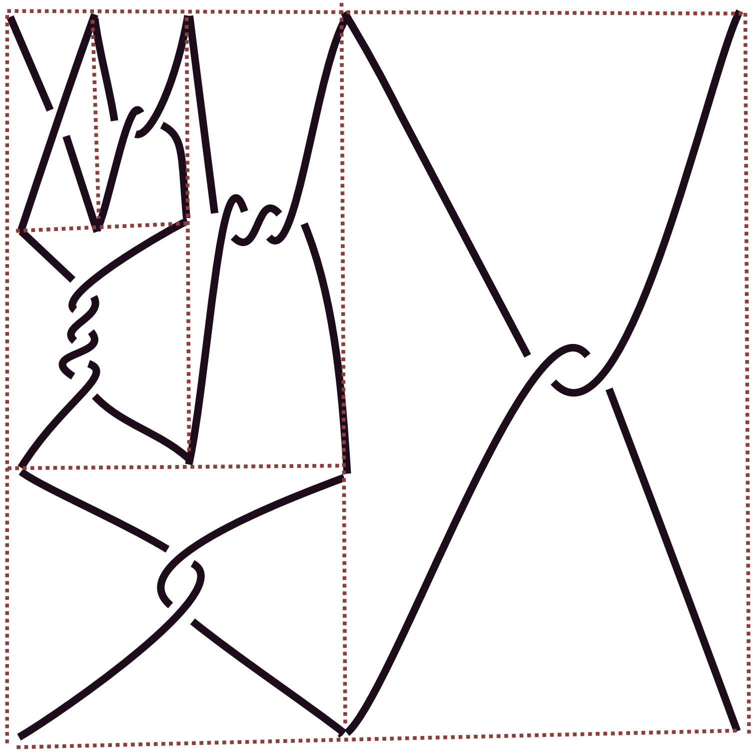
It is not at all obvious even what the various knot polynomials are actually saying!

Current algorithms for telling if a knot is the unknot take exponential time- even computing the Jones polynomial is NP.

A large subject







And knots in higher dimension