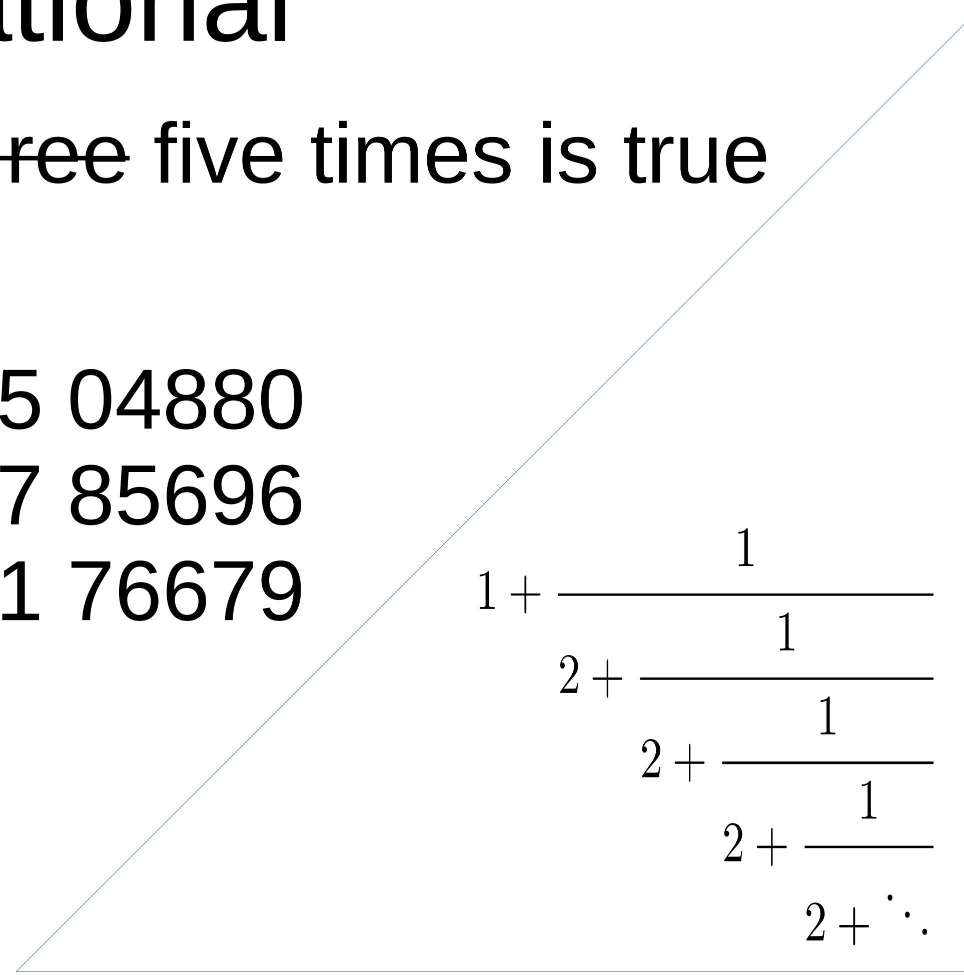
The Square Root of Two is Irrational

Or what I tell you ~~three~~ five times is true

1.41421 35623 73095 04880 16887 24209 69807 85696 71875 37694 80731 76679 73799…



$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$$

Source

I was very lucky to find the following article while preparing this talk:

Extreme Proofs 1: The irrrationality of √2
By John H Conway and Joseph Shipman
in The Best Writing in Mathematics 2014

Unique Factorization

If $a^2 = 2b^2$

Then each prime factor in a^2 occurs an even number of times, but 2 occurs an odd number of times in $2b^2$

Requires that we can't have something like

$p_1 p_2 \neq p_3 p_4 p_5$

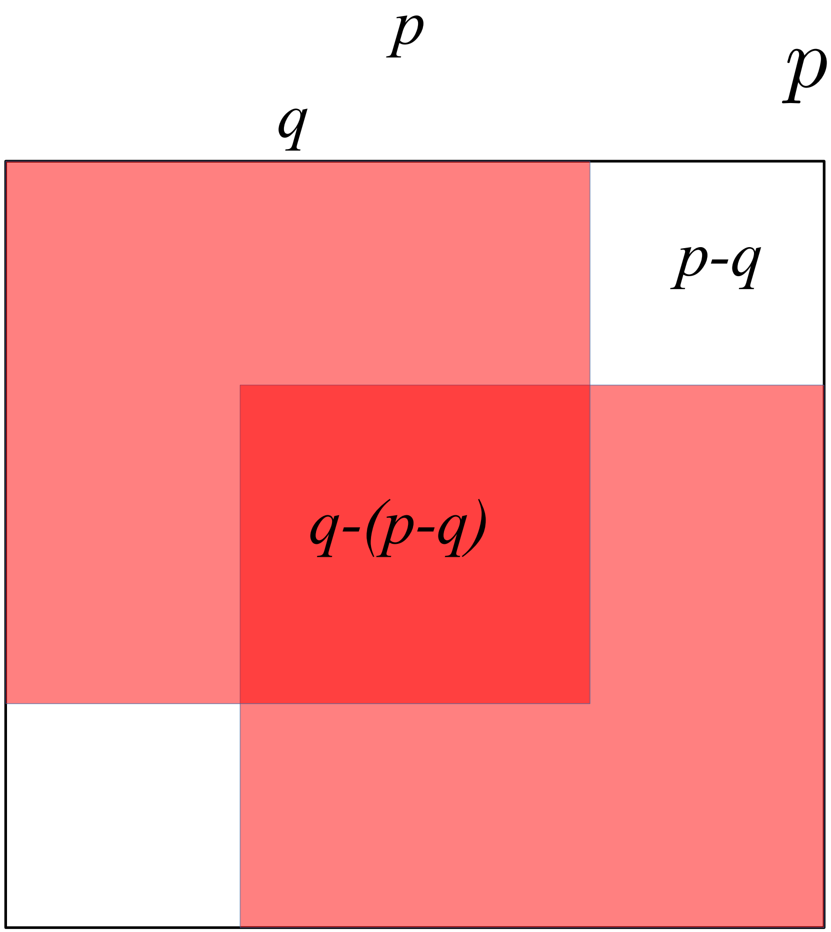
which is practically what we have to prove

Requires a far harder theorem in its proof

So we'll not count this one

Tennenbaum's 'Covering' Proof

$$p^2 = 2q^2$$

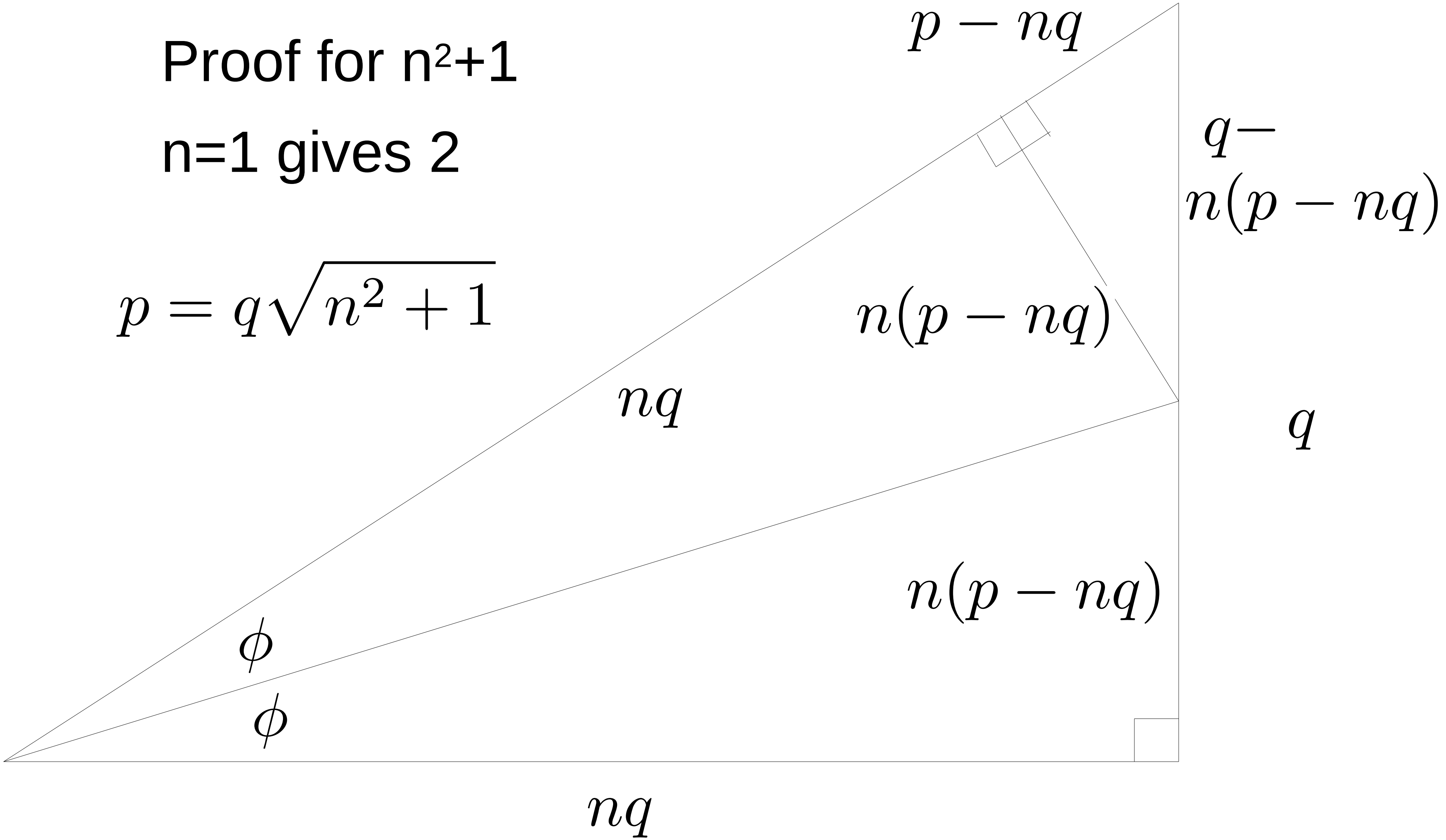


Folding Proof

Proof for n²+1

n=1 gives 2

$$p = q\sqrt{n^2+1}$$



Even/Odd Proof

Probably the simplest proof.

Aristotle wrote about it.

Suppose $\sqrt{2} = p/q$ in lowest terms

=> $p^2 = 2q^2$

=> p even and q odd

Let $p=2r$ then $4r^2 = 2q^2$

=> $2r^2 = q^2$

So q is even – giving a contradiction

Modulo 8 Proof

Theodorus of Cyrene (~5th century BC) drew figures to show the square roots of non squares up to 17 were incommensurable.

N^2 =0, 1 or 4 modulo 8

Fro this we can show that no odd non-square can have a rational square root provided N ≠ 1 mod 8

May have been the basis of the proof by Theodorus as the language doesn't really say if 17 is included.

However too complex and just a derivative of the even odd proof – so won't count this one.

Reciprocation Proof

Suppose √2 is rational *P/Q* in lowest terms

$$\sqrt{2} = P/Q = 2/\sqrt{2} = 2Q/P$$

They have fractional parts since they are not integers and they are equal

$$\sqrt{2} - 1 = q/Q = p/P \qquad p < P \;\; \& \;\; q < Q$$

Therefore -

$$p/q = P/Q$$

Analytic Proof

$$\sqrt{2} - 1 < 1$$

$$(\sqrt{2} - 1)^n \to 0$$

$$(\sqrt{2} - 1)(a\sqrt{2} + b) = (b - a)\sqrt{2} + (2a - b)$$

So if √2-1 is a rational p/q
the minimum it can be is 1/q

Constructive proof

Proof avoiding the law of the excluded middle

$2q^2$-p^2 cannot be zero as they have different powers of 2. This is practically a proof normally using a contradiction

$$|2q^2 - p^2| \geq 1$$

$$\left|\sqrt{2} - \frac{p}{q}\right| = \frac{|2q^2 - p^2|}{q^2(\sqrt{2} + p/q)} \geq \frac{1}{q^2(\sqrt{2} + p/q)} \geq \frac{1}{3q^2}$$

So no rational can be closer to the square root of 2 than that – but the first step requires the idea of doubly even or oddly even so it requires some more proof.